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The Geometry of the Reachability Cone for Linear Discrete–time Systems

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1 Introduction and definitions

In this paper we study the geometrical properties of the set of reachable states x_k of a single input discrete–time LTI system of the form:

$$x_{k+1} = F\,x_k + g\,u_k \qquad\qquad k = 0, 1, \ldots \tag{1}$$

with $F \in I\!R^{n\times n}$, $g \in I\!R^n$ when the input function u_k is nonnegative for all times k. It is worth noting that nonnegativity of the input implies that the reachable set is a convex *cone*[1]. In fact, the set of states reachable in k steps can be written as[2]

$$\mathcal{R}_k(F,g) = \left\{ x : x = \sum_{i=0}^{k-1} F^{k-i-1} g\,u(i),\ \ u(i) \geq 0 \right\} =$$

$$\text{cone}\left(g, Fg, \ldots F^{k-1}g\right)$$

In what follows, we will consider the geometrical properties of the *reachable set* $\mathcal{R}(F,g)$ of a reachable pair (F,g) defined as

$$\mathcal{R}(F,g) = \text{cl}\left\{ \sum_{k=1}^{\infty} \mathcal{R}_k(F,g) \right\} = \text{cl}\left\{\text{cone}\left(g, Fg, F^2g \ldots\right)\right\} \tag{2}$$

where the sum of two cones, as proved in [7], Theorem 3.8, coincides with the set of all finite nonnegative combinations of vectors belonging to the two cones. The reachability set $\mathcal{R}(F,g) \subseteq I\!R^n$ can be written as

$$\mathcal{R}(F,g) = \mathcal{S}(F,g) \oplus \mathcal{K}(F,g)$$

where $\mathcal{S}(F,g)$ is the maximal linear subspace contained in $\mathcal{R}(F,g)$ and $\mathcal{K}(F,g)$ is a proper[3] cone contained in the subspace $\mathcal{S}(F,g)^c$ complementary to $\mathcal{S}(F,g)$ in $I\!R^n$.

[1] A set $\mathcal{K} \subset \mathbb{R}^m$ is said to be a *cone* provided that $\alpha\mathcal{K} \subseteq \mathcal{K}$ for all $\alpha \geq 0$.

[2] The notation cone$(v_1, \ldots, v_M)$ indicates the convex cone consisting of all nonnegative linear combinations of vectors $v_1, \ldots, v_M$, with M possibly infinite.

[3] If a cone $\mathcal{K} \subseteq I\!R^m$ contains an open ball of $I\!R^m$ then it is said to be *solid* and if $\mathcal{K} \cap \{-\mathcal{K}\} = \{0\}$ it is said to be *pointed*. A cone which is closed, convex, solid and pointed is said a *proper cone*.

The problem of characterizing the geometrical properties of the reachable set $\mathcal{R}(F,g)$ of linear system has been studied by Evans and Murthy in [5] for discrete–time systems and by Brammer, Saperstone and Yorke, and Ohta *et al.* in [3, 9, 6] for continuous–time systems. Evans and Murthy, Brammer, and Saperstone and Yorke derived conditions for complete controllability, *i.e.* $\mathcal{R}(F,g) = \mathcal{S}(F,g) = I\!R^n$, while Ohta *et al.* provided a simple formula to evaluate the dimension of the largest reachable subspace, *i.e.* the dimension of $\mathcal{S}(F,g)$.

In this paper we deal with discrete–time systems and first we present a simple formula to evaluate the dimension of the largest reachable subspace $\mathcal{S}(F,g)$ analogous to that presented in [6] for continuous–time systems. Secondly, we provide conditions for polyhedrality[4] of the cone $\mathcal{K}(F,g)$.

We begin with some definitions. Given a square matrix F, $p_F(\lambda)$ is its characteristic polynomial, σ_F denotes the set of its eigenvalues and $\deg \lambda_i$, with $\lambda_i \in \sigma_F$, is the size of the largest block containing λ_i in the Jordan canonical form of F. If the matrix F has at least one nonnegative real eigenvalue, then ω_F equals the maximal nonnegative real eigenvalue of F; otherwise $\omega_F = 0$. Using the above definitions, the set σ_F can be partitioned in the following disjoint subsets:

$$\begin{aligned}
\sigma_F^{(1)} &= \{\lambda_i \in \sigma_F : |\lambda_i| > \omega_F\} \\
\sigma_F^{(2)} &= \{\lambda_i \in \sigma_F : |\lambda_i| = \omega_F \text{ and } \deg \lambda_i > \deg \omega_F\} \\
\sigma_F^{(3)} &= \{\lambda_i \in \sigma_F : |\lambda_i| = \omega_F \text{ and } \deg \lambda_i \leq \deg \omega_F\} \\
\sigma_F^{(4)} &= \{\lambda_i \in \sigma_F : |\lambda_i| < \omega_F \}
\end{aligned}$$

so that $\sigma_F := \sigma_F^{(0)} = \sigma_F^{(1)} \cup \sigma_F^{(2)} \cup \sigma_F^{(3)} \cup \sigma_F^{(4)}$. Moreover, given a set of eigenvalues $\sigma_F^{(k)}$, we define

$$\rho(\sigma_F^{(k)}) = \max_{\lambda_i \in \sigma_F^{(k)}} \{|\lambda_i|\}$$

[4] A cone $\mathcal{K}$ is said to be *polyhedral* if it is expressible as the intersection of a finite family of closed half-spaces.

and every eigenvalue $\lambda_i \in \sigma_F^{(k)}$ such that $|\lambda_i| = \rho(\sigma_F^{(k)})$ will be called a *dominant eigenvalue* of $\sigma_F^{(k)}$.

If F is nonderogatory, then *w.l.o.g.* we can assume the matrix to be in following pseudo–Jordan form

$$F = \left(\begin{array}{cc|ccc} J\left(\sigma_F^{(1)}\right) & 0 & 0 & 0 & 0 \\ 0 & J'\left(\sigma_F^{(2)}\right) & * & 0 & 0 \\ \hline 0 & 0 & J''\left(\sigma_F^{(2)}\right) & 0 & 0 \\ 0 & 0 & 0 & J\left(\sigma_F^{(3)}\right) & 0 \\ 0 & 0 & 0 & 0 & J\left(\sigma_F^{(4)}\right) \end{array}\right) \tag{3}$$

$$= \begin{pmatrix} A' & * \\ 0 & A \end{pmatrix}, \quad g = \begin{pmatrix} b' \\ b \end{pmatrix}$$

where

$$\begin{aligned} J\left(\sigma_F^{(k)}\right) &= \operatorname{diag}_{\lambda_i \in \sigma_F^{(k)}} \left(J_{\deg \lambda_i}\left(\lambda_i\right)\right) \qquad k = 1, 3, 4 \\ J'\left(\sigma_F^{(2)}\right) &= \operatorname{diag}_{\lambda_i \in \sigma_F^{(2)}} \left(J_{\deg \lambda_i - \deg \omega_F}\left(\lambda_i\right)\right) \\ J''\left(\sigma_F^{(2)}\right) &= \operatorname{diag}_{\lambda_i \in \sigma_F^{(2)}} \left(J_{\deg \omega_F}\left(\lambda_i\right)\right) \end{aligned}$$

and $J_k(\lambda)$ is a $k \times k$ upper triangular matrix of the form

$$J_k(\lambda) = \begin{pmatrix} \lambda & 1 & & & 0 \\ & \lambda & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ 0 & & & & \lambda \end{pmatrix}$$

The dimension of the matrix A' is

$$\mu = \sum_{\lambda_i \in \sigma_F^{(1)}} \deg \lambda_i + \sum_{\lambda_i \in \sigma_F^{(2)}} \left(\deg \lambda_i - \deg \omega_F\right)$$

and that of A is

$$\chi = n - \mu = \sum_{\lambda_i \in \sigma_F^{(2)}} \deg \omega_F + \sum_{\lambda_i \in \sigma_F^{(3)} \cup \sigma_F^{(4)}} \deg \lambda_i$$

where summation over the empty set is considered to be zero.