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“Il significato ultimo del pensiero matematico risiede secondo me nell’idea di una sottile complessa armonia fra tutte le realtà visibili e invisibili.”

Ennio De Giorgi[1]

[1] Da: “Una conversazione con Ennio De Giorgi” in: E. De Giorgi, *Anche la scienza ha bisogno di sognare*, a cura di F. Bassani, A. Marino, C. Sbordone, editrice Plus (Pisa), pag. 166. La conversazione era già stata pubblicata in traduzione inglese nel volume “Partial differential equations and the calculus of variations, essays in honor of Ennio De Giorgi”, a cura di F. Colombini, A. Marino, L. Modica, S. Spagnolo, Birkhauser (1989).

Calculus of Variations: topics from the mathematical heritage of E. De Giorgi

edited by D. Pallara

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Preface

As soon as the project of the publication of a volume dedicated to the memory of Ennio De Giorgi was conceived, two things appeared immediately to be quite clear, trying to keep the volume in a reasonable size. One was the impossibility to cover the whole range of De Giorgi's mathematical interests, the other one the impossibility to ask of a significant number of his pupils and friends to contribute to the volume. Too large has been the spectrum of the topics where De Giorgi gave important contributions, from partial differential equations and the calculus of variations to functional analysis, (geometric) measure theory, differential geometry, up to the foundations of mathematics, not to mention other fields where his point of view had some influence, though (strictly speaking) he gave no personal contribution, such as probability theory, mathematical physics, numerical analysis. I shall not even try to give an account of De Giorgi's mathematical work, which has been authoritatively described in the *Bollettino dell'Unione Matematica Italiana*, **2-B** (1999), 1-31, by L. Ambrosio, G. Dal Maso, M. Forti, M. Miranda, S. Spagnolo, and will be easily accessible in the forthcoming edition of his selected papers.

For the above reasons, I confined myself to the field that De Giorgi perhaps liked most (and which is the closest to my interests), and to only a few contributions, due to Authors who were closely associated with De Giorgi, and are still closely associated with each other.

It seems to me that the resulting collection reflects quite well, albeit in the restricted perspective just mentioned, several aspects of the mathematical flavour of De Giorgi's ideas and several examples of their usefulness in applications. All the papers collected in the present volume witness the presence and the fertility of De Giorgi's ideas in the Calculus of variations. Most of them center around the space of functions of bounded variation, which have been settled in the present form in the fifties, with fundamental contributions by De Giorgi. His interest in their fine properties started from the geometric properties of sets with finite perimeter and eventually led him to single out the space of *special* BV functions, where several classes of variational problems can be fruitfully studied, basically *free discontinuity problems*. In all the papers of this volume functions of bounded variation, their subspaces and their generalisations are strongly used, sometimes in a very abstract framework (see Ambrosio-Miranda Jr-Pallara), some other times in order to deal with very concrete problems in mathematical physics (elasticity, see Colombo-Tomarelli, and fracture theory, see Dal Maso-Francfort-Toader) or image segmentation. In this connection, the paper by Carriero-Leaci-Tomarelli discusses some theoretical issues on second order segmentation models and the paper by Carriero-Farina-Sgura presents some numerical applications of both first and second order variational models, based on an approximation in the framework of De Giorgi's Γ-convergence theory. The paper by Buttazzo-Stepanov deals with a class of variational problems that includes, among others, urban planning, irrigation, shape optimisation problems, and the paper of Cianchi-Fusco is closer to the theory of sets of finite perimeter and the related symmetrization properties of minimisers of several variational functionals (again, the prototype is De Giorgi's celebrated result on the isoperimetric property of the sphere in the class of finite perimeter sets).

Finally, I want to express my deep gratefulness to all the friends who accepted to contribute to this volume, and to my friends of the Seconda Università di Napoli, who gave me the possibility of honouring, with this volume, the memory of one of the most influential mathematicians of our time. We still miss very much Ennio De Giorgi, a great scientist and a great man.

Lecce, 8 June, 2004

Diego Pallara

Special Functions of Bounded Variation in Doubling Metric Measure Spaces

L. Ambrosio, M. Miranda Jr, D. Pallara

Contents

1. Introduction

In the last few years there has been an increasing interest in the analysis in metric spaces: a short list far from being exhaustive includes the papers [12], [14], [15], [35], [6], [38], [28], [30], [29], [31] and the books [37], [36], [7], with quite succesful attempts to understand the fine properties of Lipschitz, Sobolev and BV functions and the theory of sets of finite perimeter.

In this paper we consider a metric measure space (X, d, μ) with μ doubling and we assume that a Poincaré inequality with upper gradients is valid in this space. This framework is quite general and includes for instance all compact Riemannian manifolds and all Carnot–Carathéodory spaces. In this setting, the theory of BV functions and the study of the fine properties of sets of finite perimeter have been studied in the papers [3], [4], [41].

Here we extend to this setting the theory of SBV functions, the so-called special functions of bounded variation, whose derivative is made by a "volume" energy and a "surface" energy, see [5] as a reference book on this topic. In the Euclidean case, one of the most succesful applications of the SBV theory has been the rigorous analysis of the Mumford–Shah functional; here we prove the basic compactness theorem of SBV functions and we investigate some natural extensions of the Mumford–Shah functional to a metric setting.

The plan of the paper is the following. Section 2 has mostly an expository nature and contains all basic examples of metric measure spaces with a Poincaré inequality (among them CC spaces and groups and the weighted BV spaces of [8]). Section 3 has the same nature as well and deals with basic facts of the Euclidean theory of special functions of bounded variation and the Mumford–Shah functional.

In Section 4 we recall the basic facts of the theory of BV functions and the fine properties of sets of finite perimeter: in particular, following [4], we identify a codimension 1 or "surface" measure $\mathcal{S}^h$ by applying the Carathédory construction to the function

$$h(\overline{B}_\varrho(x)) := \frac{\mu(\overline{B}_\varrho(x))}{\varrho}.$$

It turns out that the perimeter measure $P(E, \cdot)$ is representable in terms of $\mathcal{S}^h$ and it is concentrated on $\partial^* E$, the essential boundary of E. Denoting by θ_E

the function such that

$$P(E,B) = \int_{B\cap\partial^* E} \theta_E \, d\mathcal{S}^h$$

we improve some results of [4] by showing that θ_E is bounded not only from below, but also from above, by universal constants (actually the bound from above involves the doubling constant only).

In Section 5 we define SBV functions in the same spirit as the original paper [23], by requiring that the total variation is the sum of a measure absolutely continuous with respect to μ and a measure absolutely continuous with respect to $\mathcal{S}^h$ and concentrated on a set σ-finite with respect to $\mathcal{S}^h$. Moreover, using the coarea formula of [41] we estabilish a chain rule for the computation of $|D(\psi\circ u)|$, with ψ of class C^1, strictly increasing and Lipschitz, and we use this chain rule to show, adapting the argument in [2] (see also [1]), that, as in the Euclidean theory, $u\in SBV(X)$ and $\mathcal{S}^h(S_u)$ is finite if and only if

$$|D(\psi\circ u)| \le \psi'(u)a\mu + (\operatorname{osc}\psi)\nu \qquad \forall\psi$$

for some $a\in L^1(\mu)$ and some finite measure ν in X. This immediately leads to the closure property of SBV functions, as in the Euclidean theory.

Regarding possible definitions of a "Mumford–Shah" energy in this setting, the basic difficulty is that at this level of generality no lower semicontinuous surface energy is known, besides the perimeter. Therefore, since the jump set can be represented through unions of intersections of essential boundaries, see Proposition 5.1, it is natural to define a surface energy of the jump set by glueing all perimeter measures, i.e., defining a measure σ concentrated on S_u such that

$$\sigma(B) = P(\{u<t\},B) \qquad \text{for any Borel set } B\subset S_u\cap\partial^*\{u<t\}.$$

However this construction seems to work only under an additional technical "locality" condition on the space: whenever $E\subset F$ are sets of finite perimeter, it should happen that

$$\theta_E = \theta_F \qquad \mathcal{S}^h\text{-a.e. on } \partial^* E\cap\partial^* F.$$

Under this locality condition we are able in Section 6 to define σ and to prove the lower semicontinuity of the Mumford–Shah type energy

$$\int_X |Gu|^p \, d\mu + \alpha \int_X |u-g|^q \, d\mu + \beta\sigma(S_u)$$

(here $|Gu|$ is the density of $|Du|$ with respect to μ, $g \in L^\infty(X)$, $p > 1$, $q > 0$). This, in conjunction with the closure of SBV functions, leads to the existence of minimisers for the functional.

In Section 7 we show that our class of "local" spaces includes all Carnot groups of step 2 (thanks to the rectifiability result proved in [31]) and all spaces induced by a continuous and strong A_∞ weight. In this way we recover previous results by Song and Yang [45], Citti, Manfredini and Sarti [17] in the Heisenberg group and by Franchi and Baldi [10] in weighted spaces.

Notation Given a metric space (X,d), we denote by $B_\varrho(x)$ the open ball and by $\overline{B}_\varrho(x)$ the closed ball centred at $x \in X$ with radius $\varrho > 0$. With the notation $B(X)$ we mean the collection of all closed balls of X, and with $\mathcal{B}(X)$ the collection of all Borel sets. If $B = B_\varrho(x)$ is any ball, we denote with $2B$ the ball with the same centre x as B and with the double radius, i.e., $2B = B_{2\varrho}(x)$. Given $F \subset X$ μ-measurable, the symbol $\mu \llcorner F$ denotes the restriction measure, i.e., $\mu \llcorner F(E) = \mu(E \cap F)$ for any μ-measurable set E. The N-dimensional Lebesgue and the k-dimensional Hausdorff and spherical Hausdorff measures in $\mathbb{R}^N$ are denoted by $\mathcal{L}^N$, $\mathcal{H}^k$, $\mathcal{S}^k$, respectively. In the metric space (X,d) the k-dimensional spherical Hausdorff measure is denoted by $\mathcal{S}^k_d$.

2. Doubling metric spaces with a Poincaré inequality

In this section we give the basic definitions we use in this article, together with the main consequences. The framework is given by a complete metric space (X,d) with a given positive measure μ defined on the Borel sets $\mathcal{B}(X)$ of X which we assume for simplicity to be finite. The main assumptions we make on the metric measure space (X,d,μ) are:

1. the measure μ is doubling;

2. the space (X, d, μ) supports a Poincaré inequality.

Let us comment these assumptions and present some examples.

Definition 2.1. *The measure μ is said to be* doubling *if there exists a constant $c > 0$ such that the following condition holds for every closed ball $\overline{B}_\varrho(x) \in B(X)$*

$$\mu(\overline{B}_{2\varrho}(x)) \leq c\mu(\overline{B}_\varrho(x)). \tag{2.1}$$

We say that μ is asymptotically doubling *if*

$$\limsup_{\varrho \downarrow 0} \frac{\mu(B_{2\varrho}(x))}{\mu(B_\varrho(x))} < +\infty \qquad \forall\ x \in X.$$

We shall denote by C_D the least constant that satisfies condition (2.1), i.e., we define

$$C_D = \sup_{B \in B(X)} \frac{\mu(2B)}{\mu(B)}. \tag{2.2}$$

Let us see some examples of doubling measures in metric spaces (X, d, μ), starting from the simplest ones.

Example 2.1 -

1. If we take $X = \mathbb{R}^N$, $d(x, y) = |x - y|$ the Euclidean metric and $\mu = \mathcal{L}^N$ the Lebesgue measure, then it is easy to verify that $(\mathbb{R}^N, |\cdot|, \mathcal{L}^N)$ is a doubling metric measure space with $C_D = 2^N$.

2. Let $X = (M, g)$ be a complete Riemannian manifolds of dimension N and μ is the canonical measure associated with the metric tensor g. Then, if the Ricci curvature is nonnegative, from [16, Proposition 4.1] it follows that μ is doubling with $C_D = 2^N$.

3. In this example we show that the dimension of the doubling metric measure space (X, d, μ) is not necessarily constant; in fact, if we take $X = [-1, 0] \times [-1, 1] \cup [0, 1] \times \{0\}$, d the Euclidean metric and $\mu = \mathcal{L}^2 \llcorner X + \mathcal{H}^1 \llcorner [0, 1] \times \{0\}$, then μ is doubling with $C_D = 4$. Spaces with constant dimensions are briefly discussed in Example 5 below.

4. We give here an abstract construction of Cantor sets taken from [19] and [43]; this construction shows that any Cantor-type set has a structure of doubling metric measure space. Fix a finite set F of at least two elements and consider the set of sequences of elements of F

$$F^\infty = \{x = (x_i)_{i\in\mathbb{N}} : x_i \in F\}.$$

Fixed $a \in (0,1)$, let us define the distance

$$d_a(x,y) = \begin{cases} 0 & \text{if } x = y \\ a^j & \text{if } x_i = y_i \text{ for } i < j \text{ and } x_j \neq y_j. \end{cases}$$

The measure is constructed as follows. Take the uniformly distributed probability measure ν on F, and define the measure μ on F^∞ as the product measure of ν infinitely many times; it turns out that

$$\mu(B_{a^j}(x)) = \frac{1}{k^j}$$

where k is the cardinality of F. With this construction we have that (F^∞, d_a, μ) is a doubling metric measure space with dimension s given by the equation

$$a^s = \frac{1}{k}.$$

The case $a = 1$ still gives a metric measure space, but it is not doubling; moreover, it is possible to prove that if F has exactly two elements and $a = 1/3$, then the previous contruction defines a space that is bilipschitz equivalent to the standard Cantor set.

5. Let us discuss an example that falls into the class discussed in the preceding item, the Sierpinski carpet, using its classical construction, rather than the previous one. Let $Q = [0,1] \times [0,1] \subset \mathbb{R}^2$ be the unit square, divide Q in nine equal squares of sidelength $1/3$ and remove the central one. In this way we obtain a set Q_1 which is the union of 8 squares of sidelength $1/3$; repeating this procedure in each square we get a sequence of sets Q_j consisting of 8^j squares of sidelength $1/3^j$. Finally, we define the Sierpinski carpet to be $S = \bigcap Q_j$; we notice that if d is the distance on $\mathbb{R}^2$ given by

$$d((x,y),(x_0,y_0)) = |x - x_0| + |y - y_0|,$$

then (S, d) is a complete geodesic metric space. The measure μ on this space is given by the weak* limit of the uniform probability measures μ_j concentrated on Q_j. This measure is nothing but that the Hausdorff measure of dimension s, where s is defined by the equation

$$3^s = 8;$$

it is not difficult to prove that μ is a doubling measure (it is the weak* limit of uniformly doubling measures).

As we can deduce from the previous examples, the doubling condition gives only an upper bound on the dimension of the space X; moreover, if (X, d, μ) is a doubling metric measure space with a given doubling constant C_D, we can add to the space X other sets with lower Hausdorff dimension. This can be clarified with the following consequences of the doubling condition.

Remark 2.1 -

1. There exists a lower bound for the density of the space X; more precisely, if we set $s = \log_2 C_D$, then

$$\frac{\mu(B_\varrho(x))}{\mu(B_R(y))} \geq \frac{1}{C_D^2}\left(\frac{\varrho}{R}\right)^s, \quad \forall\ 0 < \varrho \leq R < +\infty, \quad x, y \in X. \tag{2.3}$$

   This means that in some sense the number $s = \log_2 C_D$ defines a dimension on X; it is called the homogeneus dimension of X. We point out that this is not the topological dimension of X (it can be greater), and it depends on μ and on the metric d. As we shall see, if we change the metric d, then the homogeneus dimensions may change as well.

2. The balls are totally bounded, hence closed balls are compact. Then, the notion of doubling metric measure space is intrinsically finite-dimensional; this implies that it is not possible to put doubling measures on infinite dimensional spaces. In particular Hilbert spaces with Gaussian measures cannot be doubling, not even locally.

3. The measure μ is finite if and only if the diameter of X is finite. In fact, if $d = \mathrm{diam}(X)$ is finite, then trivially, taking an arbitrary ball B_ϱ with

$\varrho > 0$, for $n > d/\varrho$ we get

$$\mu(X) \le \mu(B_{n\varrho}) \le C_D^n \mu(B_\varrho).$$

Conversely, assume that $\mathrm{diam}(X) = +\infty$. Then, fix a point $y \in X$ and two radii ϱ, R with $0 < \varrho < R/2$. Then, for infinitely many $n \in \mathbb{N}$ there is a ball $B_\varrho(x_n)$ contained in the annulus $B_{2^n R}(y) \setminus B_{2^{n-1}R}(y)$ with the property that any point $x \in X$ lies at most in two of such balls. From (2.3) we know that $\mu(B_\varrho(x_n)) \ge C_D^{-2}(\varrho/R)^s \mu(B_R(y))$ for every n, whence $\mu(X) \ge 1/2 \sum_n \mu(B_\varrho(x_n)) = +\infty$. As a consequence, even in a finite-dimensional space, probability measures with strictly positive densities are never doubling.

4. The doubling condition implies the Lebesgue differentiation Theorem and the Maximal Theorem; then, for a given function $u \in L^1_{loc}(X, \mu)$, it is possible to talk about Lebesgue points, and it is possible to define the maximal operator and obtain the same continuity properties from $L^p(X, \mu)$ to $L^p(X, \mu)$ as in the Euclidean case (see [37, Theorem 2.2]).

   More generally, the Lebesgue differentation Theorem allows to compute the density of the absolutely continuous part of a measure ν with respect to a doubling (or asymptotically doubling) measure μ, even when a Besicovich type Theorem doesn't hold. In fact, writing $\nu = f\mu + \nu^s$, with ν^s singular with respect to μ, it is possible to compute

$$f(x) = \frac{d\nu}{d\mu}(x) = \lim_{\varrho \downarrow 0} \frac{\nu(B_\varrho(x))}{\mu(B_\varrho(x))}$$

   for μ-almost every $x \in X$ (see for instance [37, Theorem 1.8]).

We give now the definition of Poincaré inequality; in order to do that, we introduce a notion of gradient of a function defined on a metric space. This is strictly related to the problem of the definition of Sobolev spaces on metric spaces. Hajłasz defined the Sobolev space $W^{1,p}(X, \mu)$, $p > 1$, as the set of the functions $u \in L^p(X, \mu)$ such that there exists a function $g \in L^p(X, \mu)$, $g \ge 0$, such that

$$|u(x) - u(y)| \le d(x, y)(g(x) + g(y)). \tag{2.4}$$

In the Euclidean setting, i.e., in the case $(X,d,\mu)=(\mathbb{R}^N,|\cdot|,\mathcal{L}^N)$, relation (2.4) is satisfied if $u\in W^{1,p}(\mathbb{R}^N)$ with $g=c(N)M(|\nabla u|)$, where $M(|\nabla u|)$ is the maximal function of the gradient of u and $c(N)$ is a constant depending only upon the dimension N. Clearly, the continuity of the maximal operator implies the equivalence of the definitions of Sobolev spaces, but (2.4) gives a notion of gradient that is not pointwise; to overcome this problem, Hajłasz and Koskela gave another definition of gradient for a function, the upper gradient.

Definition 2.2 (Upper Gradient)**.** *Given a continuous function $u:X\to\mathbb{R}$, we say that $g:X\to[0,+\infty]$ is an upper gradient for u if for every $x,y\in X$ and for every rectifiable curve γ joining x to y the inequality*

$$|u(x)-u(y)|\leq\int_\gamma g$$

holds.

To be sure that the notion of upper gradient is consistent, we note that, for every Lipschitz continuous function $u:X\to\mathbb{R}$, the function

$$|\nabla u|(x)=\liminf_{\varrho\downarrow 0}\sup_{y\in B_\varrho(x)}\frac{|u(x)-u(y)|}{\varrho} \tag{2.5}$$

is an upper gradient for u (see [15]).

Let us come to discuss the (weak) Poincaré Inequality.

Definition 2.3. *We say that the space (X,d,μ) supports a* weak Poincaré inequality *if there exist constants $c_P>0$, $\lambda>1$, such that for every continuous function $u:X\to\mathbb{R}$ and for every upper gradient g the inequality*

$$\int_B|u-u_B|\,d\mu\leq\varrho\cdot c_P\int_{\lambda B}g\,d\mu \tag{2.6}$$

holds for every ball $B\in B(X)$ of radius ϱ, where u_B indicates the average of u over B and λB is the ball with the same centre as B and radius $\lambda\varrho$.

In the paper [39] it is shown that if a weak Poincaré inequality holds for every Lipschitz continuous function u and with $g=|\nabla u|$, then the space X supports a weak Poincaré inequality. This means that in order to prove that

X supports a weak Poincaré inequality it suffices to verify (2.6) for u Lipschitz continuous and $g = |\nabla u|$.

Let us notice that not every doubling metric space supports a Poincaré inequality. In fact, if $X = A \cup B$ with $A, B \subset \mathbb{R}^N$ bounded open sets with $\mathrm{dist}(A, B) > 0$ and $\mu(A), \mu(B) > 0$, $d = |\cdot|$ and $\mu = \mathcal{L}^N$, then $u = \chi_A$ is Lipschitz continuous on X, $|\nabla u| = 0$ but

$$\int_X |u - u_X| \, d\mu = \mu(A) > 0.$$

Then in some sense the Poincaré inequality implies some kind of connectedness and even something more, i.e., the so-called *quasi-convexity* of the space X. In fact, if the space (X, d, μ) is a doubling space and supports a Poincaré inequality, then (see [43]) the space is *quasi-convex*, in the sense that there exists a constant $c > 0$ such that if δ is the geodesic distance induced by d on X, then

$$d(x, y) \le \delta(x, y) \le c d(x, y), \quad \forall x, y \in X.$$

We recall that the geodesic distance is defined as

$$\delta(x, y) = \inf \{ length(\gamma) : \gamma \text{ is a rectifiable curve joining } x \text{ and } y \}.$$

It is possible to prove (see e.g. [37, Theorem 4.18]) that if a doubling metric satisfies a weak Poincaré inequality, then Poincaré inequality (2.6) holds with $\lambda = 1$ and with the geodesic metric.

Inequality (2.6) can be stated in an equivalent way as follows:

$$\min_{c \in \mathbb{R}} \int_B |u - c| \, d\mu \le \varrho \cdot c_P \int_{\lambda B} g \, d\mu,$$

which has the advantage of being invariant under bilipschitz mapping.

It is also possible to prove (see [15]) that in a doubling metric space supporting a Poincaré inequality, a μ-almost everywhere differentiability result holds for Lipschitz continuous functions. Indeed, if u is Lipschitz continuous, then for μ-a.e. $x \in X$ we have

$$\begin{aligned} \liminf_{\varrho \downarrow 0} \sup_{y \in B_\varrho(x)} \frac{|u(x) - u(y)|}{\varrho} &= \limsup_{\varrho \downarrow 0} \sup_{y \in B_\varrho(x)} \frac{|u(x) - u(y)|}{\varrho} \\ &= \lim_{\varrho \downarrow 0} \sup_{y \in B_\varrho(x)} \frac{|u(x) - u(y)|}{\varrho}. \end{aligned}$$

We give now a short list of examples of doubling metric spaces supporting a Poincaré inequality, and also an example of a quasi-convex doubling metric space which doesn't support a Poincaré inequality.

Example 2.2 -

1. As we have seen, finite-dimensional complete Riemannian manifolds with positive Ricci curvature are doubling; moreover, Buser's inequality (see [13, Theorem 1.2]) implies Poincaré inequality.

2. Let us consider the Euclidean space $\mathbb{R}^N$ endowed with the measure $\mu = \omega\mathcal{L}^N$, with a strongly A_∞ (in the sense of [18]) nonnegative weight $\omega \in L^1_{loc}(\mathbb{R}^N)$. Recall that ω is an A_∞ weight if for every $\varepsilon > 0$ there is $\delta > 0$ such that for any ball $B \subset \mathbb{R}^N$ and $E \subset B$ the implication

$$\mathcal{L}^N(E) \leq \delta\mathcal{L}^N(B) \qquad \Longrightarrow \qquad \mu(E) \leq \varepsilon\mu(B)$$

holds. Then, the measure μ is a doubling measure. Moreover, it is possible to define the quasi-distance

$$\delta(x,y) := \mu\left(B_{x,y}\right)^{1/N},$$

where $B_{x,y}$ is the (Euclidean) ball with diameter $|x-y|$ containing x and y, and ω is a strong-A_∞ weight if the distance δ is equivalent to the geodesic distance d_ω associated with the Riemannian metric $\omega^{1/N}ds$. In this case the functions that are Lipschitz continuous in the Euclidean metric are also Lipschitz continuous with respect to the distance d_ω and if we compute the gradient $|\nabla_\omega u|$ of a Lipschitz function u using (2.5), we obtain

$$|\nabla_\omega u|(x) = \omega(x)^{-1/N}|\nabla u|(x). \tag{2.7}$$

It is also known that the doubling metric measure space $(\mathbb{R}^N, d_\omega, \mu)$ supports a Poincaré inequality (see [18], [27], [28]).

3. In $\mathbb{R}^N$, given the vector fields $X = (X_1, \ldots, X_k)$, $k < N$, X verifies Hörmander's condition if there is an integer p such that the family of

commutators of the X_i up to the length p span $\mathbb{R}^N$ at every point. Moreover, we say that a Lipschitz path $\gamma : [0,T] \to \mathbb{R}^N$ is admissible for X, or also *horizontal*, if there exist measurable functions $a_1, \dots, a_k : [0,T] \to \mathbb{R}$, with $a_1^2(t) + \dots + a_k^2(t) \le 1$ and

$$\gamma'(t) = \sum_{i=1}^{k} a_i(t) X_i(\gamma(t)) \qquad \text{for a.e. } t \in [0,T]. \tag{2.8}$$

Then, it is possible to define the Carnot-Carathéodory metric by setting

$$d(x,y) = \inf \left\{ T : \exists\, \gamma : [0,T] \to \mathbb{R}^N \text{ as in (2.8), } \gamma(0) = x, \gamma(T) = y \right\};$$

if no such curve exists, then we set $d(x,y) = +\infty$. It is possible to prove (Chow Theorem, see [33, Theorem 0.4]) that if the vector fields X satisfy Hörmander's condition, then every two points can be joined with an admissible curve of finite length. With the definition of the Carnot-Carathéodory distance, it is possible to prove that if u is a Lipschitz function with respect to d, then the function

$$|Xu| = \sqrt{|X_1 u|^2 + \dots |X_k u|^2}$$

is the minimal upper gradient for u (see [36, Section 11.2]). An example of Carnot–Carathéodory space is given by the Grušin plane; it is $\mathbb{R}^2$ with the vector fields $X_1(x,y) = (1,0)$ and $X_2(x,y) = (0,x)$. It is easily seen that $[X_1, X_2] = (0,1)$, and then Hörmander's condition is satisfied and the Carnot-Carathéodory distance is a metric (the admissible curves are those which are vertical when passing through the y-axis). Carnot groups are a special case of Carnot-Carathéodory spaces. In fact, the underlying space is endowed with a group structure, the measure is invariant under the translation group (Haar measure) and the vector fields X are obtained by fixing k tangent vectors at 0 (the identity of the group) that satisfy Hörmander's condition and extending them to all other points in such a way to be left invariant under the group action.

4. Important particular examples of Carnot groups are Heisenberg groups $\mathbb{H}^N$, given by $\mathbb{H}^N = \mathbb{C}^N \times \mathbb{R}$, whose points are denoted by $P = [z,t]$, with

the group operation

$$[z_1, t_1] \cdot [z_2, t_2] = [z_1 + z_2, t_1 + t_2 + 2Im(z_1\overline{z_2})], \tag{2.9}$$

where $z_1, z_2 \in \mathbb{C}^N,\ t_1, t_2 \in \mathbb{R}$ and we also write $z = (x, y)$ with $x, y \in \mathbb{R}^N$. The inverse element is $P^{-1} = [-z, -t]$. The distance d_C on $\mathbb{H}^N$ is given by the Carnot-Carathéodory metric induced by the left-invariant vector fields

$$\begin{cases} X_j(z,t) = X_j(x,y,t) = \partial_{x_j} + 2y_j\partial_t \\ Y_j(z,t) = Y_j(x,y,t) = \partial_{y_j} - 2x_j\partial_t, \end{cases}$$

$j = 1, \ldots, N$, which satisfy Hörmander's condition, the only non-trivial commutator relation being $[X_j, Y_j] = -4\partial_t$, $j = 1, \ldots, N$, and the measure is the Lebesgue measure $\mathcal{L}^{2N+1}$. Then, the space $(\mathbb{H}^N, d_C, \mathcal{L}^{2N+1})$ is a doubling metric measure space supporting a Poincaré inequality. Notice that the distance d_C is globally equivalent to that induced by the homogeneous norm $\|Q\| = \|[z,t]\|_\infty = \max\{|z|, |t|^{1/2}\}$ through $d(P_1, P_2) = \|P_2^{-1} \cdot P_1\|_\infty$.

5. Let us recall that a Borel regular measure μ in (X, d) is called s-regular if there exist two constant $c, C > 0$ such that for every $B_\varrho(x) \in B(X)$, $c\varrho^s \le \mu(B_\varrho(x)) \le C\varrho^s$. If μ is s-regular then X is called an s-regular or Ahlfors regular space. These spaces are particular examples of doubling metric measure spaces in which there is also an upper bound for the density of the measure; this implies that there is also a control from below on the dimension, and then there is a well defined notion of dimension that is constant on the whole space. In particular, the measure is equivalent to the s-dimensional Hausdorff measure; we have seen examples of regular spaces in Examples 2.1 (1), (4), (5) and Examples 2.2 (3), (4). It is worth noticing that Laakso ([40]) constructed for *every* real number $s \ge 1$ an example of a metric measure space (X, d, μ) with μ an s-regular measure supporting a Poincaré inequality.

6. It is not true that every doubling metric space which is quasi-convex supports a Poincaré inequality; the counterexample is given by the Sierpinski carpet S defined in Example 2.1 (5). To see that it doesn't support a

Poincaré inequality, it suffices to consider the sequence of Lipschitz functions

$$u_n(x,y) = \begin{cases} 1 & x \le \frac{1}{2} - \frac{1}{n} \\ -\frac{n}{2}x + \frac{n}{4} + \frac{1}{2} & |x - \frac{1}{2}| \le \frac{1}{n} \\ 0 & x \ge \frac{1}{2} + \frac{1}{n}. \end{cases}$$

Clearly $u_n \to \chi_{S \cap [0,1/2] \times [0,1]}$, but

$$\int_S |\nabla u_n| \, d\mu \to 0.$$

Using the terminology of perimeter that we shall see in the sequel, this example shows that in this case the set $A = S \cap [0,1/2] \times [0,1]$ has null perimeter in S, but $\mu(A) = \mu(S \setminus A) = 1/2$.

There is a wide literature regarding Sobolev functions in doubling metric measure spaces supporting a weak Poincaré inequality. We recall that these definitions can be given also without the requirement of doubling condition and Poincaré inequality, but with these two conditions a large number of properties true in the Euclidean case are still valid. We recall in particular the following.

1. The Sobolev embedding, i.e., the continuous embedding of $W^{1,p}(X,\mu)$ in $L^{p^*}(X,\mu)$ if $p < s$, with $p^* = \frac{sp}{s-p}$.
2. The Hölder continuity of Sobolev functions with high summability: if $p > s$ then functions of $W^{1,p}(X,\mu)$ are Hölder continuous.
3. The Rellich-Kondrachov compact embedding theorem, i.e., the compact embedding of $W^{1,p}(X,\mu)$ in $L^q(X,\mu)$ when X is bounded (or, equivalently, $\mu(X) < +\infty$), for every $p \ge 1$, $1 \le q < p^*$.

3. Free discontinuity problems

Free discontinuity problems are variational problems where the functional to be minimised consists of terms describing a volume energy, in general represented by an integral with respect to the volume measure, and terms describing a surface energy, which is concentrated on a lower dimensional set. Typically, competitors are pairs (K, u), where K is closed and the admissible function u is required to be regular outside K and the volume terms depend upon the derivatives of the competitor functions. The term that indicates this class of problems was introduced by E. De Giorgi in [22], with emphasis on the fact that the two involved variables can in some sense be coupled. In fact, the basic idea to embody this class of problems in the well-established stream of direct methods of the calculus of variations has been to relax the problem by introducing a suitable enlarged class of functions that are admissible for the volume energy, but such that the lower dimensional unknown set that carries the surface energy could be interpreted as the set of the discontinuities of the competitor function. With the above notation, we may think of u as a (possibly) discontinuous function defined on the whole space, denoting by K the set where u is discontinuous. If we fix an open set $\Omega \subset \mathbb{R}^N$, and we think of K as a surface (i.e., a $(N-1)$-dimensional set) in Ω, we are quickly led to the space of functions of bounded variation BV, which can be discontinuous (in the measure theoretic sense) precisely along $(N-1)$-dimensional sets. Indeed, starting from [23], the class that has proved to be suitable to deal with these problems has been the class of *special* BV functions (and its variants), whose gradient can be split in a volume term and a $(N-1)$-dimensional term. Let us recall that, given an open set $\Omega \subset \mathbb{R}^N$, a function u belongs to $SBV(\Omega)$ if $u \in L^1(\Omega)$ and its distributional gradient Du is a measure such that

$$|Du|(B) = \int_B |\nabla u|\, dx + \int_{B\cap S_u} |u^\vee(x) - u^\wedge(x)|\, d\mathcal{H}^{N-1}$$

for any Borel set B. Here, ∇u denotes now the density of the absolutely continuous part of the distributional derivative of u with respect to the Lebesgue measure, or, equivalently, the approximate gradient of u, and $u^\vee$, $u^\wedge$, S_u are defined in the present more general setting in Definition 5.1 below.

In general doubling metric measure spaces, as we have seen, there are some generalisations of the notion of gradient that can naturally enter in a volume energy term, but there is nothing similar concerning higher order derivatives, hence of course only first order problems can be considered (see, however, [6] for higher order problems in some special cases). On the other hand, there is no natural notion of "surface" or even of "measure of codimension 1" to be used in place of Hausdorff measure, used in the classical contexts. Let us recall the prototype of free discontinuity problem, i.e., the Mumford-Shah functional; given an open set $\Omega \subset \mathbb{R}^N$ and a function $g \in L^\infty(\Omega)$, for $p > 1$ and $q > 0$ it can be defined as follows

$$\mathcal{F}(K,u) = \int_\Omega |\nabla u|^p \, dx + \alpha \int_\Omega |u-g|^q \, dx + \beta \mathcal{H}^{N-1}(K \cap \Omega)$$

for every closed set $K \subset \mathbb{R}^N$ and $u \in C^1(\Omega \setminus K)$, and can be relaxed in $SBV(\Omega)$ by setting

$$F(u) = \int_\Omega |\nabla u|^p \, dx + \int_\Omega |u-g|^q \, dx + \mathcal{H}^{N-1}(S_u).$$

The existence of a minimising pair for $\mathcal{F}$ has been proved starting form the minimisation of F in $SBV(\Omega)$; after that, the regularity theory for the SBV minimisers started, and a number of properties of minimisers has been found. We refer to [5] for a detailed (even though not up-to-date, by now) account of the treatment of Mumford-Shah problem and for a description of the general framework of free discontinuity problems in $\mathbb{R}^N$.

Mumford–Shah functional has been recently studied in some generalised settings, precisely weighted spaces as in Example 2.2 (2), see [10], and the Heisenberg groups, see [17]. In both cases, generalisations of the classical compactness and lower semicontinuity results are in fact available (see also [44], [45]), relying upon arguments that closely follow the Euclidean techniques. We shall come back to these examples in Section 7.

4. BV functions and perimeters

In this section we give the definition of BV functions and recall their main properties. Among all the characterisations of BV functions available in the Euclidean case, the one which has proved to be the most suitable in the present general setting is based upon a relaxation procedure starting from $W^{1,1}(X,\mu)$ functions. We recall that, due to a work of Cheeger, this method gives an alternative definition of the Sobolev spaces $W^{1,p}(X,\mu)$ for $p > 1$.

Given a function $u \in L^1_{loc}(X,\mu)$, we define

$$|Du|(X) = \inf_{(u_n)_n \in \mathcal{A}_u(X)} \left\{ \liminf_{n\to+\infty} \int_X |\nabla u_n|\, d\mu \right\},$$

where

$$\mathcal{A}_u(X) = \left\{ (u_n) : (u_n) \subset \mathrm{Lip}_{loc}(X), u_n \stackrel{L^1_{loc}(X,\mu)}{\longrightarrow} u \right\}.$$

Then we say that a function $u \in L^1(X,\mu)$ has bounded variation, $u \in BV(X)$, if $|Du|(X) < +\infty$. Moreover given a set $E \subset X$, we say that E has finite perimeter if $|D\chi_E|(X) < +\infty$.

In a similar way, given any open subset $A \subset X$, we may define $|Du|(A)$ (just substitute X with A in the previous definition), and obtain that $|Du|$ is the restriction to the open subsets of a Borel regular measure (see [41, Theorem 3.4]). If $u = \chi_E$ then we set $P(E,A) = |D\chi_E|(A)$.

If $u \in \mathrm{Lip}(X)$, then $u \in BV_{loc}(X)$, and then its total variation measure is well defined. Of course, since $|Du| << \mu$, we have $|Du| = |Gu|\mu$ for some function $|Gu|$, and it is possible to see that there is $c \geq 1$ such that

$$|Gu| \leq |\nabla u| \leq c|Gu|,$$

but, to our knowledge, it is not known whether the following equality

$$|Du|(A) = \int_A |\nabla u|(x)\, d\mu(x)$$

is true for every Lipschitz continuous function.

Remark 4.1 - Since the space X supports a Poincaré inequality, then there are $c_P > 0, \lambda > 1$ such that for every function $u \in BV(X)$ and for every ball $B_\varrho(x) \subset X$, we have

$$\int_{B_\varrho(x)} |u(y) - u_{B_\varrho(x)}|\, d\mu(y) \leq \varrho \cdot c_P |Du|(B_{\lambda\varrho}(x)).$$

Moreover, $u \in BV(X)$ if and only if there exist $\lambda > 1$ and a finite positive measure ν on X such that

$$\int_{B_\varrho(x)} |u(y) - u_{B_\varrho(x)}|\, d\mu(y) \leq \varrho\nu(B_{\lambda\varrho}(x)), \quad \forall x, \varrho.$$

Let us also recall the following isoperimetric inequalities. The first one is

$$\min\{\mu(B_\varrho(x)\cap E),\mu(B_\varrho(x)\setminus E)\} \tag{4.1}$$
$$\leq c_I\left(\frac{\varrho^s}{\mu(B_\varrho(x))}\right)^{\frac{1}{s-1}} P(E,B_{\lambda\varrho}(x))^{\frac{s}{s-1}}$$

and is a direct consequence the Sobolev embedding. The second one requires an estimate on the measure of $B_\varrho(x)\setminus E$ and can be easily deduced from the first one. If $\gamma\in]0,1/2]$ and

$$\min\Big\{\mu(B_\varrho(x)\cap E)),\mu(B_\varrho(x)\setminus E)\Big\}\geq \gamma\mu(B_\varrho(x))$$

then

$$\max\{\mu(B_\varrho(x)\cap E),\mu(B_\varrho(x)\setminus E)\} \tag{4.2}$$
$$\leq c_\gamma\left(\frac{\varrho^s}{\mu(B_\varrho(x))}\right)^{\frac{1}{s-1}} P(E,B_{\lambda\varrho}(x))^{\frac{s}{s-1}},$$

where $c_\gamma = c_I\frac{1-\gamma}{\gamma}$. Indeed, if $\mu(B_\varrho(x)\cap E)\geq\mu(B_\varrho(x)\setminus E)$, then

$$\begin{aligned}\mu(B_\varrho(x)\cap E) &= \mu(B_\varrho(x))-\mu(B_\varrho(x)\setminus E)\leq\left(\frac{1-\gamma}{\gamma}\right)\mu(B_\varrho(x)\setminus E)\\ &\leq \left(\frac{1-\gamma}{\gamma}\right)c_I\left(\frac{\varrho^s}{\mu(B_\varrho(x))}\right)^{\frac{1}{s-1}} P(E,B_{\lambda\varrho}(x))^{\frac{s}{s-1}}\end{aligned}$$

by (4.1), and so (4.2) is proved. If $\mu(B_\varrho(x)\setminus E)\geq\mu(B_\varrho(x)\setminus E)$ the argument is similar.

We note that there is a strict relationship between function of bounded variation and sets of finite perimeter; in fact, as in the classical case, the hypograph of a BV function is a set with (locally) finite perimeter. In $X\times\mathbb{R}$ we use the distance $\tilde{d}((x,t),(y,s))=\max\{d(x,y),|t-s|\}$.

Proposition 4.1. *Let $u\in L^1(X)$ be a nonnegative function and define*

$$H(u)=\{(x,t)\in X\times\mathbb{R}: 0\leq t\leq u(x)\}\,;$$

then, $u \in BV(X)$ if and only if $H(u)$ is a set of locally finite perimeter in $X \times \mathbb{R}$; moreover, if $\mu(X) < +\infty$, the following inequalities hold

$$|Du|(X) \le P(H(u), X \times \mathbb{R}) \le |Du|(X) + \mu(X).$$

PROOF – Let us suppose that u belongs to $BV(X)$; then by definition there exists a sequence $(u_n)_n \subset \mathrm{Lip}_{loc}(X)$ such that

$$u_n \xrightarrow{L^1_{loc}} u, \qquad \int_X |\nabla u_n(x)|\, d\mu(x) \to |Du|(X).$$

We define a new sequence of Lipschitz functions $\psi_n : X \times \mathbb{R} \to \mathbb{R}$ by setting

$$\psi_n(x,t) = \begin{cases} 1 & t < u_n(x) \\ \\ n(t - u_n(x)) & u(x) \le t \le u_n(x) + 1/n \\ \\ 0 & t > u(x) + 1/n. \end{cases}$$

Then we have that ψ_n is a sequence of Lipschitz functions converging to $\chi_{H(u)}$ and with

$$|\nabla \psi_n(x,t)| \le n(1 + |\nabla u_n(x)|)$$

where $u_n(x) \le t \le u_n(x) + 1/n$ and

$$|\nabla \psi_n(x,t)| = 0$$

elsewhere. Then, if $\nu = \mu \times \mathcal{L}^1$ is the product measure,

$$\int_{X \times \mathbb{R}} |\nabla \psi_n(x,t)|\, d\nu(x,t) \le \mu(X) + \int_X |\nabla u_n(x)|\, d\mu(x),$$

and then we get

$$P(H(u), X \times \mathbb{R}) \le \mu(X) + |Du|(X).$$

On the other hand, if ψ_n is a sequence of Lipschitz functions converging to the characteristic function of the set $H(u)$, then defining

$$u_n(x) = \int_0^{+\infty} \psi_n(x,t)\, dt$$

we obtain a sequence of Lipschitz functions converging to u in L^1 and such that

$$\int_X |\nabla u_n|(x)\, d\mu \le \int_{X\times\mathbb{R}} |\nabla \psi_n(x,t)|\, d\nu(x,t).$$

□

A much more precise relationship between sets of finite perimeter and function of bounded variation is given by the following Theorem, which relates the total variation of a function u with the perimeters of the sublevels of u.

Theorem 4.1 (Coarea Formula). *For every $u \in BV(X)$ and every Borel set $A \subset X$, we have*

$$|Du|(A) = \int_{-\infty}^{+\infty} P(\{u > t\}, A)\, dt.$$

We point out that, by taking $u(x) = d(x, x_0)$, the coarea formula shows that almost every ball $B_\varrho(x)$ in X has finite perimeter, but *a priori* this is not true for every ball.

In the classical case, it is well known that the perimeter measure can be concentrated on a small subset of the topological boundary; this leads to define first the *essential* or *measure theoretic* boundary

$$\partial^* E = \{x \in X : \Theta^*(E,x) > 0,\ \Theta^*(X \setminus E, x) > 0\},$$

where $\Theta^*(E,x)$ and $\Theta_*(E,x)$ are the upper and lower densities of E at x:

$$\Theta^*(E,x) := \limsup_{\varrho\downarrow 0} \frac{\mu(E\cap B_\varrho(x))}{\mu(B_\varrho(x))}, \qquad \Theta_*(E,x) := \liminf_{\varrho\downarrow 0} \frac{\mu(E\cap B_\varrho(x))}{\mu(B_\varrho(x))},$$

and, in the Euclidean case only, the generalised inner normal to a set of finite perimeter E at $|D\chi_E|$-a.e. x, given by

$$\nu_E(x) := \lim_{\varrho\downarrow 0} \frac{D\chi_E(B_\varrho(x))}{|D\chi_E|(B_\varrho(x))}$$

and the reduced boundary of E:

$$\mathcal{F}E = \left\{x \in \mathbb{R}^N : \exists\ \nu_E(x) \text{ and } |\nu_E(x)| = 1\right\}.$$

Then (see e.g. [5, Theorems 3.59, 3.61]) it turns out that the set $\mathcal{F}E$ is countably $\mathcal{H}^{N-1}$-rectifiable and the following equalities holds

$$P(E,B) = \mathcal{H}^{N-1}(B \cap \mathcal{F}E), \qquad \mathcal{H}^{N-1}(\partial^* E \setminus \mathcal{F}E) = 0. \tag{4.3}$$

In a metric setting, it is not possible to define the normal direction and the reduced boundary, and only the essential boundary of E makes sense; moreover, since the metric space has only a homogeneous dimension, and in general the Hausdorff dimension may change locally, we cannot use the Hausdorff measure $\mathcal{H}^{N-1}$. Hence, we proceed by defining another Hausdorff-like measure, as in [4].

Let us define the function $h : B(X) \to [0,+\infty]$ as (see also [32], where the same function appears in a similar context)

$$h(\overline{B}_\varrho(x)) = \frac{\mu(\overline{B}_\varrho(x))}{\varrho}; \tag{4.4}$$

due to the doubling condition on the measure μ, the function h turns out to be a doubling function, i.e., $h(\overline{B}_{2\varrho}(x)) \le (C_D/2)h(\overline{B}_\varrho(x))$ for every $x \in X$, $\varrho > 0$ (where C_D is the constant in (2.2)). Then, using the Carathéodory construction, we may define the generalised Hausdorff spherical measure $\mathcal{S}^h$ as

$$\mathcal{S}^h(A) = \liminf_{\varrho \downarrow 0} \left\{ \sum_{i=0}^{\infty} h(B_i) : B_i \in B(X),\ A \subset \bigcup_{i=0}^{\infty} B_i,\ \mathrm{diam}\,(B_i) \le \varrho \right\},$$

which was introduced in [4]. As a consequence of the doubling property of h, a Vitali-type covering theorem holds and this in turn implies the following density estimate (see [4, Theorem 2.1], [26, 2.10.19]):

$$\limsup_{\varrho \downarrow 0} \frac{\nu(B_\varrho(x))}{h(\overline{B}_\varrho(x))} \ge t \quad \forall\, x \in B \qquad \Longrightarrow \qquad \nu(B) \ge t\mathcal{S}^h(B) \tag{4.5}$$

for any locally finite measure ν in X and any $B \in \mathcal{B}(X)$. Notice that the estimate from above

$$\limsup_{\varrho \downarrow 0} \frac{\nu(B_\varrho(x))}{h(\overline{B}_\varrho(x))} \le t \quad \forall\, x \in B \qquad \Longrightarrow \qquad \nu(B) \le t\mathcal{S}^h(B) \tag{4.6}$$

is always true for Hausdorff spherical measures.

Let us show that the measure $P(E,\cdot)$ is absolutely continuous with respect to the measure $\mathcal{S}^h$. More precisely, the following representation formula holds (see [4, Theorems 5.3, 5.4]).

Theorem 4.2. *Given a set of finite perimeter E, the measure $P(E,\cdot)$ is concentrated on the set $\Sigma_\gamma \subset \partial^* E$ defined by*

$$\Sigma_\gamma = \left\{ x \in X : \liminf_{\varrho \downarrow 0} \min\left\{ \frac{\mu(E \cap B_\varrho(x))}{\mu(B_\varrho(x))}, \frac{\mu((X \setminus E) \cap B_\varrho(x))}{\mu(B_\varrho(x))} \right\} \geq \gamma \right\} \tag{4.7}$$

with $\gamma > 0$ depending only upon C_D and c_I. Moreover, $\mathcal{S}^h(\partial^* E \setminus \Sigma_\gamma) = 0$, $\mathcal{S}^h(\partial^* E) < +\infty$ and there are $\alpha > 0$, depending only upon C_D and c_I, and a Borel function $\theta_E : X \to [\alpha, +\infty[$ such that

$$P(E, B) = \int_{B \cap \partial^* E} \theta_E(x)\, d\mathcal{S}^h(x), \quad \forall B \in \mathcal{B}(X). \tag{4.8}$$

Finally, the perimeter measure is asymptotically doubling, i.e., for $P(E,\cdot)$-a.e. $x \in X$ we have

$$\limsup_{\varrho \downarrow 0} \frac{P(E, B_{2\varrho}(x))}{P(E, B_\varrho(x))} < +\infty.$$

As a consequence of the asymptotic doubling property we have the following differentiation property (see [26, 2.8.17, 2.9.7]): for $\nu = \lambda P(E,\cdot)$ we have

$$\lim_{\varrho \downarrow 0} \frac{\nu(\overline{B}_\varrho(x))}{P(E, \overline{B}_\varrho(x))} = \lambda(x) \qquad \text{for } P(E,\cdot)\text{-a.e. } x \in X. \tag{4.9}$$

We can improve the above result by showing that in fact the density function θ_E is bounded from above by a universal constant. In order to prove this result, we need the following variant of Proposition 5.7 in [4]. We present a complete proof for reader's convenience.

Proposition 4.2. *Let $\gamma \in]0, 1/2[$ and $M > 1$ be given. Then, for $P(E,\cdot)$-a.e. $x \in X$ there exists $\varrho_x > 0$ such that, for a.e. $\varrho \in]0, \varrho_x[$, the volume bound*

$$\min\left(\mu(B_\varrho(x) \cap E), \mu(B_\varrho(x) \setminus E)\right) \geq \gamma \mu(B_\varrho(x))$$

implies the estimate

$$P(E, B_\varrho(x)) \leq M P(E \setminus B_\varrho(x), \partial B_\varrho(x)). \tag{4.10}$$

Proof – We can consider the family $\mathcal{G}$ of all closed balls of X satisfying

1. $\mu(\partial B_\varrho(x)) = 0$ and $P(E, \partial B_\varrho(x)) = 0$;
2. $\min\left(\mu(E \cap B_\varrho(x)), \mu(B_\varrho(x) \setminus E)\right) \geq \gamma\mu(B_\varrho(x))$;
3. $P(E, B_\varrho(x)) > MP(E \setminus B_\varrho(x), \partial B_\varrho(x))$.

Note that condition 1. is satisfied, for every fixed $x \in X$, for almost every ball. We set $B = \cap_j B_j$, where B_j is the set of all points $x \in X$ such that the set

$$\{\varrho \in]0, 2^{-j}[: \overline{B}_\varrho(x) \in \mathcal{G})\}$$

has positive measure. The set B is a Borel set (see [4, Proposition 5.7]) and what we have to prove is that $P(E, B) = 0$, or equivalently $P(E, K) = 0$ for every compact set $K \subset B$. Let $\varepsilon > 0$ be fixed and define the family

$$\mathcal{F} = \{\overline{B}_\varrho(x) \in \mathcal{G} : x \in K, \varrho \in]0, \varepsilon[\}.$$

By construction and the relative isoperimetric inequality (4.2), we have that for every $\overline{B}_\varrho(x) \in \mathcal{F}$

$$P(E, B_{\lambda\varrho}(x)) \geq \left(\frac{\gamma}{c_\gamma}\right)^{\frac{s-1}{s}} h(\overline{B}_\varrho(x)).$$

We may apply Vitali covering Theorem [4, Theorem 2.1] to the family $\mathcal{F}$ getting a disjoint at most countable family $(\overline{B}_{\varrho_i}(x_i))_{i\in I} \subset \mathcal{F}$ containing $\mathcal{S}^h$ almost all K (by absolute continuity, the same family contains $P(E, \cdot)$ almost all of K). If we set $A_\varepsilon = \cup_{i\in I} B_{\varrho_i}(x_i)$, we have

$$\begin{aligned}
\mu(A_\varepsilon) &= \sum_{i\in I} \varrho_i h(\overline{B}_{\varrho_i}(x_i)) \\
&\leq \varepsilon \left(\frac{\gamma}{c_\gamma}\right)^{\frac{s-1}{s}} \sum_{i\in I} P(E, \overline{B}_{\lambda\varrho_i}(x_i)) \\
&\leq \varepsilon \left(\frac{\gamma}{c_\gamma}\right)^{\frac{s-1}{s}} P(E, X).
\end{aligned}$$

Then, if $\varepsilon \to 0$, $\mu(A_\varepsilon) \to 0$; in addition, A_ε satisfies $P(E, K \setminus A_\varepsilon) = 0$. Hence, if $J \subset I$ is a finite set and $A_J = \cup_{i\in J} B_{\varrho_i}(x_i)$, we get

$$\begin{aligned} P(E \setminus A_J, X) &= P(E \setminus A_J, X \setminus A_J) \\ &= P(E \setminus A_J, X \setminus \overline{A}_J) + P(E \setminus A_J, \partial A_J) \\ &\leq P(E, X \setminus \overline{A}_J) + \sum_{i\in J} P(E \setminus B_{\varrho_i}(x_i), \partial B_{\varrho_i}(x_i)) \\ &< P(E, X \setminus K) + \frac{1}{M} P(E, A_\varepsilon). \end{aligned}$$

Then by letting $J \to I$ and $\varepsilon \to 0$, by the lower semicontinuity of the perimeter we get

$$P(E, X) \leq P(E, X \setminus K) + \frac{1}{M} P(E, K),$$

hence $P(E, K) = 0$. □

We are now in a position to prove the announced upper bound for the density θ_E.

Theorem 4.3. *Let E be a set of finite perimeter E, and let θ_E be the function in Theorem 4.2. Then, $\theta_E \leq C_D$, where C_D is the doubling constant in (2.2).*

Proof – For every $x \in X$ and $\varrho > 0$, let us define $m_E(x, \varrho) = \mu(E \cap B_\varrho(x))$, denoting by $m'_E(x, \varrho)$ its derivative with respect to ϱ; set further $E^c = X \setminus E$. Applying Proposition 4.2 with $M > 1$ fixed and γ given by Theorem 4.2, we can compute, for $P(E, \cdot)$-a.e. $x \in X$ and for a.e. $\varrho \in]0, \varrho_x[$:

$$\begin{aligned} 2P(E, B_\varrho(x)) &= P(E, B_\varrho(x)) + P(E^c, B_\varrho(x)) \\ &\leq MP(E \setminus B_\varrho(x), \partial B_\varrho(x)) + MP(E^c \setminus B_\varrho(x), \partial B_\varrho(x)) \\ &\leq Mm'_E(x, \varrho) + Mm'_{E^c}(x, \varrho) = 2M \frac{d}{d\varrho} \mu(B_\varrho(x)), \end{aligned}$$

whence $P(E, B_\varrho(x)) \leq M \frac{d}{d\varrho} \mu(B_\varrho(x))$ and then

$$\begin{aligned} P(E, B_\varrho(x)) &\leq \frac{1}{\varrho} \int_\varrho^{2\varrho} P(E, B_r(x))\, dr \leq \frac{M}{\varrho} \mu(B_{2\varrho}(x)) \\ &\leq C_D M \frac{\mu(B_\varrho(x))}{\varrho} = C_D M h(B_\varrho(x)). \end{aligned}$$

As a consequence,

$$\limsup_{\varrho\downarrow 0}\frac{P(E,B_\varrho(x))}{h(B_\varrho(x))}\leq C_D M.$$

Since this is true for any $M>1$, we finally get, taking the limit $M\to 1$,

$$\limsup_{\varrho\downarrow 0}\frac{P(E,B_\varrho(x))}{h(B_\varrho(x))}\leq C_D.$$

By (4.8) and (4.6) we infer that $\theta_E\leq C_D$ $\mathcal{S}^h$-a.e. in X. □

5. SBV functions and the compactness theorem

In this section we develop the fine BV theory, along the lines of the Euclidean theory, and introduce the class of *special* BV functions, using the definition of SBV given in [23]. We prove a chain rule and a characterisation of SBV functions through composition originally obtained in the Euclidean setting in [2]. These results, in connection with the representation of the perimeter, will allow us to prove the closure and the compactness theorem for SBV.

First of all, let us recall the definition of upper and lower approximate limits and the related definition of S_u.

Definition 5.1 (Upper and lower approximate limits)**.** *Let $u:X\to\mathbb{R}$ be a measurable function and $x\in X$; we define the* upper *and* lower approximate limits of u at x *respectively by*

$$\begin{aligned} u^\vee(x) &:= \inf\left\{t\in\overline{\mathbb{R}}:\ \lim_{\varrho\downarrow 0}\frac{\mu(\{u>t\}\cap B_\varrho(x))}{\mu(B_\varrho(x))}=0\right\},\\ u^\wedge(x) &:= \sup\left\{t\in\overline{\mathbb{R}}:\ \lim_{\varrho\downarrow 0}\frac{\mu(\{u<t\}\cap B_\varrho(x))}{\mu(B_\varrho(x))}=0\right\}. \end{aligned}$$

If $u^\vee(x)=u^\wedge(x)$ we call their common value, denoted $\tilde u(x)$, the approximate limit of u at x*. We also set $S_u=\{x\in X:\ u^\wedge(x)<u^\vee(x)\}$.*

Notice that if $u=\chi_E$, then $S_u=\partial^*E$. If $u\in L^\infty_{loc}(X)$ and $x\notin S_u$, then

$$\lim_{\varrho\downarrow 0}\frac{1}{\mu(B_\varrho(x))}\int_{B_\varrho(x)}|u(y)-\tilde u(x)|\,d\mu(y)=0. \tag{5.1}$$

We have the following characterisation of S_u.

Proposition 5.1. *Let u belong to $L^1(X,\mu)$; then*

$$S_u = \bigcup_{t,\, s\in D,\ s\neq t} \partial^*\{u>s\}\cap\partial^*\{u>t\},$$

where $D\subset\mathbb{R}$ is any dense set. Moreover, if $u\in BV(X)$, then the dense set D can be chosen in such a way that for every $s\in D$ the set $\{u>s\}$ has finite perimeter.

PROOF – First of all we notice that

$$x\in\partial^*\{u>s\} \qquad \Longrightarrow \qquad s\in[u^\wedge(x),u^\vee(x)]. \tag{5.2}$$

Indeed, if $x\in\partial^*\{u>s\}$, we have $0<\Theta_*(\{u>s\},x)\le\Theta^*(\{u>s\},x)<1$, and then by definition of $u^\wedge$ and $u^\vee$, $u^\wedge(x)\le s\le u^\vee(x)$. In addition we have

$$x\in S_u \text{ and } s\in]u^\wedge(x),u^\vee(x)[\qquad \Longrightarrow \qquad x\in\partial^*\{u>s\}. \tag{5.3}$$

Indeed, the condition $s>u^\wedge(x)$ implies that $\Theta^*(\{u>s\},x)>0$ and the condition $s<u^\vee(x)$ implies $\Theta^*(\{u>s\},x)>0$, so that $x\in\partial^*\{u>s\}$.

Now, if $x\in S_u$ then, from (5.3), $x\in\partial^*\{u>s\}\cap\partial^*\{u>t\}$ for all $s,t\in]u^\wedge(x),u^\vee(x)[$ and then

$$x\in\bigcup_{\substack{t,s\in D\\ s\neq t}}\partial^*\{u>s\}\cap\partial^*\{u>t\}.$$

Conversely, if there exist $s<t\in\mathbb{R}$ such that

$$x\in\partial^*\{u>s\}\cap\partial^*\{u>t\}$$

then, from (5.2), $u^\wedge(x)\le s<t\le u^\vee(x)$, whence $u^\wedge(x)<u^\vee(x)$ and $x\in S_u$. If in addition we have that $u\in BV(X)$, then by the coarea formula we get that almost every set $\{u>s\}$ has finite perimeter, and then the choice of D can be done in such a way that, for every $s\in D$, the set $\{u>s\}$ has finite perimeter. □

Let us now give a decomposition result for the total variation measure of a function $u\in BV(X)$: we split $|Du|$ in three parts: an absolutely continuous measure with respect to μ, the restriction to S_u, which will be represented in terms of $\mathcal{S}^h$, and the so-called Cantor part. In the following statement, θ_E is the function introduced in Theorem 4.2 for every E of finite perimeter.

Theorem 5.1. *Let $u \in BV(\Omega)$; set $|D^d u| = |Du| \llcorner (X \setminus S_u)$ and denote by $|Gu|$ the density of $|Du|$ with respect to μ. Then, $|D^d u|(B) = 0$ for every $B \in \mathcal{B}(X)$ such that $\mathcal{S}^h(B)$ is finite, and, setting for $x \in S_u$*

$$\theta_u(x) = \int_{u^\wedge(x)}^{u^\vee(x)} \theta_{\{u>t\}}(x)\,dt, \tag{5.4}$$

we have

$$|Du| = |D^d u| + \theta_u \mathcal{S}^h \llcorner S_u = |Gu|\mu + |D^c u| + \theta_u \mathcal{S}^h \llcorner S_u, \tag{5.5}$$

where the Cantor part of $|Du|$ *is defined by* $|D^c u| = |D^d u| - |Gu|\mu$.

Proof – First of all, for every $B \in \mathcal{B}(X)$, by the coarea formula and by the representation formula for the perimeter we get

$$|Du|(B) = \int_{\mathbb{R}} P(\{u>t\}, B)\,dt = \int_{\mathbb{R}} \int_{\partial^*\{u>t\}\cap B} \theta_{\{u>t\}}(x)\,d\mathcal{S}^h(x)\,dt; \tag{5.6}$$

if $B \subset S_u$, using (5.2) and (5.3) and Fubini theorem we deduce

$$|Du|(B) = \int_{B\cap S_u} \int_{u^\wedge(x)}^{u^\vee(x)} \theta_{\{u>t\}}(x)\,dt\,d\mathcal{S}^h(x)$$

On the other hand, if $B \subset X \setminus S_u$, then the measure $|Du|$ can be split into two parts, one absolutely continuous with respect to the measure μ with density $|Gu|$, and one singular with respect to μ; we call this last part the Cantor part of the measure $|Du|$, and then we can write

$$|Du|(B) = \int_B |Gu|\,d\mu + |D^c u|(B).$$

Finally, if $B \cap S_u = \emptyset$ and $\mathcal{S}^h(B) < +\infty$, then by (5.2) for every $x \in B$ there is at most one $t \in \mathbb{R}$ such that $x \in \partial^*\{u>t\}$, namely $t = \tilde{u}(x)$. Using again (5.6), Fubini theorem and Theorem 4.3 we get

$$|Du|(B) \le C_D \int_B \mathcal{L}^1(\{t \in \mathbb{R}:\ x \in \partial^*\{u>t\}\})\,d\mathcal{S}^h(x) = 0.$$

□

Notice that in $X\setminus S_u$ the value $\tilde{u}$ is well-defined, hence we can easily deduce the following corollary of the coarea formula.

Proposition 5.2. *If $u\in BV(X)$, $A\subset X\setminus S_u$ is a Borel set and g is a bounded Borel function, then*

$$\int_A g(\tilde{u})\,d|Du| = \int_{-\infty}^{+\infty}\int_A g(\tilde{u}(x))dP(\{u>t\},\cdot)\,dt.$$

In order to extend the SBV membership criterion based on the composition with regular functions which is known in the Euclidean context (see e.g. [5, Proposition 4.12]), in the next Proposition we prove a chain rule for the composition of a BV function with an increasing C^1 function. In fact, for our purposes it is sufficient to deal with the following class:

$$\Lambda := \Big\{\psi\in C^1(\mathbb{R})\cap W^{1,\infty}(\mathbb{R}):\ \exists\, I \text{ closed interval such that } \psi'(t)=0\ \forall\, t\notin I,\ \psi \text{ is strictly increasing in } I\Big\}. \tag{5.7}$$

Proposition 5.3 (Chain Rule)**.** *For every $u\in BV(X)$ and ψ in the class Λ above defined, the function $\psi\circ u$ belongs to $BV(X)$ and the following chain rule holds:*

$$|D(\psi\circ u)| = \psi'(\tilde{u})|D^d u| + \Psi(u)\mathcal{S}^h \llcorner S_u, \tag{5.8}$$

where

$$\Psi(u)(x) = \int_{u^\wedge(x)}^{u^\vee(x)} \psi'(t)\theta_{\{u>t\}}(x)\,dt. \tag{5.9}$$

Proof – Let $B\subset S_u$, $\psi\in\Lambda$ and $I=[a,b]$ as in the definition of Λ. By the coarea formula, we get, since $\{\psi(u)>t\}=X$ for $t<\psi(a)$ and $\{\psi(u)>t\}=\emptyset$ for $t>\psi(b)$,

$$|D(\psi\circ u)|(B) = \int_{\mathbb{R}} P(\{\psi(u)>t\},B)\,dt = \int_{\psi(a)}^{\psi(b)} P(\{\psi(u)>t\},B)\,dt.$$

But then, if $t\in]\psi(a),\psi(b)]$ and we set $t=\psi(s)$, we obtain $\{\psi(u)>t\}=\{u>$

$s\}$, and thus

$$\begin{aligned}|D(\psi\circ u)|(B) &= \int_a^b \psi'(s)P(\{u>s\},B)ds = \int_{\mathbb{R}} \psi'(s)P(\{u>s\},B)ds \\ &= \int_B \int_{u^\wedge(x)}^{u^\vee(x)} \psi'(s)\theta_{\{u>s\}}(x)ds\, d\mathcal{S}^h(x).\end{aligned}$$

If $B\subset X\setminus S_u$, then $x\in\partial^*\{u>t\}$ only if $t=\tilde u(x)$ (x is an approximate continuity point of u), and then arguing as before we find

$$\begin{aligned}|D(\psi\circ u)|(B) &= \int_{\mathbb{R}} P(\{\psi(u)>t\},B)\,dt = \int_{\mathbb{R}} \psi'(s)P(\{u>s\},B)ds \\ &= \int_{\mathbb{R}}\int_B \psi'(\tilde u(x))dP(\{u>s\},\cdot)ds = \int_B \psi'(\tilde u)\,d|D^d u|.\end{aligned}$$

□

We are now in a position to define the set of special function of bounded variation, in the same vein as [23], where SBV functions have been introduced for the first time in the Euclidean space $\mathbb{R}^N$.

Definition 5.2 (SBV functions)**.** *A function $u\in BV(X)$ is said to be a special function of bounded variation, $u\in SBV(X)$, if the following holds*

$$\int_X |Gu|\,d\mu = \inf\left\{|Du|(X\setminus K) : K\subset X, \mathcal{S}^h(K)<+\infty\right\}.$$

The following characterisation of SBV functions is a direct consequence of Definition 5.2 and of the decomposition Theorem 5.1.

Proposition 5.4. *Given $u\in BV(X)$, $u\in SBV(X)$ if and only if $|D^c u|=0$.*

Let us now see another, much less obvious, characterisation of SBV functions. It is the announced membership criterion, based upon the chain rule, that will be the key point of the subsequent closure theorem. Notice that for $\psi\in\Lambda$ we set $\operatorname{osc}\psi=\max\psi-\min\psi$, where the class Λ is defined in (5.7).

Theorem 5.2. *Let $u\in BV(X)$; then, u belongs to $SBV(X)$ and $\mathcal{S}^h(S_u)<+\infty$ if and only if there exist a function $a\in L^1(X,\mu)$ and a finite positive measure ν such that*

$$|D(\psi\circ u)| \le \psi'(\tilde u)a\mu + \operatorname{osc}\psi\,\nu \tag{5.10}$$