Prolongations of Tanaka structures and regular CR structures

Dmitry V. Alekseevsky and Andrea F. Spiro

Contents

1. Introduction (3).
2. Fundamental algebra of a distribution. Regular distributions of type $m$ (9).
3. Tanaka algebras $m + g^0$ and Tanaka structures (11).
5. $G$-structures, i.e. Tanaka structures of depth one (14).
6. Full prolongations of fundamental algebras and of Tanaka algebras (20).
7. Tanaka prolongation scheme for Tanaka structures (23).
8. Tanaka structures of semisimple type and their prolongations (33).
1. Introduction

In 1907 H. Poicaré proved that two generic real hypersurfaces $M$ and $M'$ in $\mathbb{C}^2$ are not biholomorphically equivalent and stated the problem of constructing a full system of invariants, which may distinguish one hypersurface from another, up to biholomorphic equivalence. E. Cartan realized that the complex structure $J$ of $\mathbb{C}^2$ induces on a real hypersurface $M \subset \mathbb{C}^2$ a very important non-trivial geometric structure, namely, the pair $(\mathcal{D}, J|_\mathcal{D})$, where $\mathcal{D}$ is the codimension one distribution, which is the maximal $J$-invariant distribution of the tangent bundle $TM$, and $J|_\mathcal{D}$ is the complex structure on $\mathcal{D}$ induced by $J$. He pointed out that Poincaré’s problem reduces to the construction of invariants for the structure $(\mathcal{D}, J|_\mathcal{D})$, which was later called CR structure.

E. Cartan solved this problem by means of his method of “moving frames”. The main goal of this method is to associate with a manifold $M$, endowed a given structure $S$, a bundle $\pi : P = P(M) \rightarrow M$ together with an absolute parallelism, i.e. an $\mathfrak{l} = \mathbb{R}^N$-valued smooth map $\mu = \mu_M : TP \rightarrow \mathfrak{l}$ so that $\mu_u : T_u P \rightarrow \mathfrak{l}$ is an isomorphism for any $u \in P$, with the following important property. Any (local) isomorphism $f : (M, S) \rightarrow (M', S')$ has a canonical lift to an isomorphism $\tilde{f} : (P(M), \mu_M) \rightarrow (P(M'), \mu_{M'})$ and, conversely, any (local) isomorphism $\psi : (P(M), \mu_M) \rightarrow (P(M'), \mu_{M'})$ is fiber preserving and it induces a (local) isomorphism $(M, S) \rightarrow (M', S')$.

If a bundle $\pi : P(M) \rightarrow M$ and an absolute parallelism $\mu_M$ with the above properties exist, then the group $\text{Aut}(M, S)$ of automorphisms of the structure $S$ is naturally isomorphic to the group $\text{Aut}(P(M), \mu_M)$ of automorphisms of the absolute parallelism $\mu_M$. This group is a Lie group of dimension less or equal to $\dim P(M)$ by Kobayashi’s theorem ([11]). Moreover, the construction of a full system of invariants for the structure $S$ on $M$ reduces to the construction of the full system of invariants of the absolute parallelism $\mu_M$ on $P(M)$ and this can be done easily (see [23], Ch. VII).

In some good cases, the absolute parallelism $\mu$ turns out to be a Cartan connection. This means that the bundle $\pi : P(M) \rightarrow M$ has a structure of a principal bundle with structure group $Q$, the vector space $\mathfrak{l} = \mathbb{R}^N$ is a Lie algebra of a group $L$, which contains $Q$ as a proper subgroup, and the absolute parallelism $\mu_M$ is a $Q$-equivariant map (with respect to adjoint action of $Q$ on $\mathfrak{l}$), which is an extension of the vertical absolute parallelism $T^\text{vert} P \rightarrow \mathfrak{q} = \text{Lie}(Q)$.
of the principal bundle $\pi : P \to M$. In this case, the action on the fiber $P_x = \pi^{-1}(x)$ of the group $\text{Aut}(M,S)_x$ of the automorphisms fixing $x \in M$, commutes with the simply transitive action of $Q$ on $P_x$. This implies that the stability group $\text{Aut}(M,S)_x$ is isomorphic to a subgroup of $Q$ and that $\dim \text{Aut}(M,S)_x \leq \dim Q + \dim M$.

To construct the desired bundle $\pi : P(M) \to M$, E. Cartan adopted in several cases the so-called prolongation procedure. This procedure has been reformulated by S. Sternberg in the framework of $G$-structures as follows. Recall that if $G \subset GL(V)$, $V = \mathbb{R}^n$ is a linear group, then a $G$-structure on an $n$-dimensional manifold $M$ is a principal subbundle $\pi : P \to M$ of the $GL(V)$-bundle $L(M)$ consisting of all coframes $u : T_xM \to V$, $x \in M$ (see also §5 in this paper). Many important geometric structures $S$ (and, amongst them, the CR structures) can be identified with an associated $G$-structures $\pi : P \to M$, where the bundle $P$ consists of all coframes which are adapted to $S$, i.e. all coframes with respect to which the structure $S$ assumes a canonical form.

The prolongation of a $G$-structure $\pi : P \to M$ is a principal $G(1)$-bundle $\pi_1 : P^{(1)} \to P$ over $P$ with abelian structure group $G(1) = g \otimes V^* \cap V \times S^2V^*$, considered as a subgroup of $GL(V + g)$, which acts trivially on $g$. The bundle $\pi_1 : P^{(1)} \to P$ is not uniquely defined: it depends on a choice of a subspace $C \subset V \otimes S^2V^*$ which has to be complementary to $\partial(g \otimes V)$. Here, we denote by $\partial : g \otimes V \to V \otimes S^2V^*$ the natural map of antisymmetrization.

If the subspace $C$ is chosen to be $G$-invariant, then the group $G$ acts on $P^{(1)}$, normalizing $G(1)$ and in such a way that the bundle $\pi_1 \overset{\text{def}}{=} \pi \circ \pi_1 : P^{(1)} \to M$ is principal $G \cdot G(1)$-bundle.

By an inductive procedure, one can define the full prolongation of a $G$-structure $\pi : P \to M$ as a tower of $G$-structures

$$\ldots \xrightarrow{\pi_{k}} P^{(k-1)} \xrightarrow{\cdots} \xrightarrow{\pi_1} P \xrightarrow{\pi} M$$

where, for any $k \geq 1$, $\pi_k : P^{(k)} \to P^{(k-1)}$ is a $G^{(k)}$-structure on $P^{(k-1)}$ with abelian structure group

$$G^{(k)} = g \otimes S^kV^* \cap V \otimes S^{k+1}V^*.$$ 

If the group $G$ has finite order $k$, i.e. if there exists an integer $k$ such that $G^{(j)} = \{e\}$ for any $j \geq k$ and $G^{(j)} \neq \{e\}$ for any $j < k$, then the bundle

$$\pi^{k-1} = \pi \circ \pi_1 \circ \cdots \circ \pi_{k-1} : P(M) \overset{\text{def}}{=} P^{(k-1)} \to M$$
is the desired bundle endowed with the absolute parallelism given by the \{e\}-structure \(\pi_k : P^{(k)} \to P^{(k-1)}\). Furthermore, if at each step, one is able to choose an appropriate complementary subspace \(C_{i-1}\) which is invariant under the group \(G_{i-1} = G \cdot G^{(1)} \cdot \ldots G^{(i-1)}\), it follows that \(\pi_{k-1} : P(M) \overset{\text{def}}{=} P^{(k-1)} \to M\) is a principal bundle with the structure group \(G_{k-1}\). If this is the case, the absolute parallelism given by the \{e\}-structure \(\pi_k : P^{(k)} \to P^{(k-1)}\) is a Cartan connection on \(P^{(k-1)}\).

Assume, for example, that \(G \subset GL(V)\) is an irreducible subgroup of finite type. Then, it is known that either \(G^{(1)} = \{e\}\) and, in this case, \(\pi_1 : P^{(1)} \to P\) is a Cartan connection, or \(G^{(1)} \neq \{e\}\) and \(G^{(2)} = \{e\}\). In this second case, it is possible to choose the complementary subspaces in such a way that \(\pi_2 : P^{(2)} \to P^{(1)}\) is a Cartan connection on the principal \(G \cdot G^{(1)}\)-bundle \(\pi_1 : P^{(1)} \to M\).

Notice that all connected irreducible linear groups \(G \subset GL(V)\) of finite type with non-trivial first prolongation are known. They correspond to the gradations of depth 1 of simple Lie algebras \(I\)

\[ I = V + g + g^{(1)} = g^{-1} + g^0 + g^1. \]

More precisely, each such gradation determines a linear group of finite type \(G\), which is the connected linear group with the Lie algebra \(\text{ad}_g| V\).

Unfortunately, a CR structure induced on a real hypersurface of \(\mathbb{C}^2\) or, more generally, of \(\mathbb{C}^n\), determines a \(G\)-structure which is of infinite type (that is, all prolongations \(G^{(k)}\) are non trivial). Due to this fact, the standard prolongation procedure does not work. Nonetheless, for the case of a Levi non-degenerate real hypersurface \(M\) of \(\mathbb{C}^2\), E. Cartan overcame this difficulty and managed to associate with \(M\) a principal bundle \(\pi : P(M) \to M\) endowed with a canonical Cartan connection (see [4], [5]). This construction was generalized by S.- S. Chern and J. Moser in [6], where they constructed a principal bundle \(\pi : P(M) \to M\) together with a canonical Cartan connection for any Levi non-degenerate hypersurface \(M\) of \(\mathbb{C}^n, n \geq 2\). J. Moser proposed also an alternative approach for the construction of a full system of invariants for a Levi non-degenerate hypersurface in \(\mathbb{C}^2\) (based on the presentation of a canonical form for the defining equation). In [6], J. Moser and S.-S. Chern extended this second approach to the case of Levi non-degenerate hypersurfaces of \(\mathbb{C}^n, n \geq 2\) (see also the reviews in [13], [3], [30], [29]).
On the other hand, another very general approach which associates a bundle \( \pi: P(M) \to M \) with a canonical absolute parallelism to a geometric structure, has been developed by N. Tanaka (see \[24\], \[25\], \[26\]). Tanaka’s approach can be applied to a very wide class of structures and, in particular, to a large family of CR structures of arbitrary codimension.

The starting point of Tanaka’s approach consists in the observation that a distribution \( D \) on a manifold \( M \) determines a filtration of each tangent space \( T_xM \)

\[
\cdots = \mathcal{D}_{-d-1}(x) = \mathcal{D}_{-d}(x) \supseteq \mathcal{D}_{-d+1}(x) \supseteq \cdots \supseteq \mathcal{D}_{-2}(x) \supseteq \mathcal{D}_{-1}(x) = \mathcal{D}_x,
\]

where each subspace \( \mathcal{D}_{-j}(x) \subset T_xM \) is spanned by the values at \( x \) of the commutators of \( j \) vector fields \( X_1, \ldots, X_j \) in \( D \). From now on, we will assume that \( \mathcal{D}_{-d}(x) = T_xM \). Then, the commutators of vector fields induce a structure of a graded Lie algebra on the graded vector space

\[
\mathfrak{m}(x) = \text{gr}(T_xM) = \mathfrak{m}^{-d}(x) + \mathfrak{m}^{-d+1}(x) + \cdots + \mathfrak{m}^{-1}(x)
\]

associated with the filtered space \( T_xM \). Note that \( \mathfrak{m}^{-1} = \mathcal{D}_x \). A distribution \( D \) is called regular distribution of depth type \( \mathfrak{m} \) and depth \( d \) if all Lie algebras \( \mathfrak{m}(x) \), \( x \in M \), are isomorphic to a given negatively graded Lie algebra \( \mathfrak{m} \) generated by \( \mathfrak{m}^{-1} \)

\[
\mathfrak{m} = \mathfrak{m}^{-d} + \mathfrak{m}^{-d+1} + \cdots + \mathfrak{m}^{-1}.
\]

The Lie algebra \( \mathfrak{m} \) is called fundamental algebra of depth \( d \).

Let \( G^0 \subset \text{Aut}(\mathfrak{m}) \) be a subgroup of the automorphism group of a fundamental algebra \( \mathfrak{m} \). The pair \((\mathfrak{m}, G^0)\) is called Tanaka pair and the associated graded Lie algebra \( \mathfrak{m} + \mathfrak{g}^0 \), where \( \mathfrak{g}^0 = \text{Lie} G^0 \), is called Tanaka algebra.

Assume now that \((\mathfrak{m}, G^0)\) is a Tanaka pair of depth \( d \) and let \( D \) be a regular distribution of type \( \mathfrak{m} \) on a manifold \( M \). A partial \( G^0 \)-structure (also called Tanaka structure) of type \((\mathfrak{m}, G^0)\) and depth \( d \) is a principal \( G^0 \)-bundle \( \pi: P \to M \) of coframes defined on the subspaces \( \mathcal{D}_x \subset T_xM, x \in M \).

Observe that immediate examples of Tanaka structures are given by the \( G^0 \)-structures, which are indeed Tanaka structures of depth one. Other interesting examples of Tanaka structures are given by regular CR structures of codimension \( r \geq 1 \).
We recall that a CR structure of codimension $r$ on a manifold $M$ consists of a pair $(\mathcal{D}, J)$, where $\mathcal{D}$ is a distribution of codimension $r$ and $J$ denotes a smooth family of complex structures $J_x : D_x \to D_x$ on the subspaces $D_x \subset T_x M$ of the distribution, such that some suitable integrability conditions are satisfied (see (4.1) in this paper). The geometry of CR structures of codimension $r$ is that of generic real submanifolds of dimension $2n - r$ in $\mathbb{C}^n$. In fact, any real submanifold of codimension $r$ in $\mathbb{C}^n$, which satisfies some suitable regularity conditions, is endowed with the maximal $J$-invariant subdistribution $\mathcal{D}$ of the tangent bundle $TM$ and with the family $J$ of complex structures $J_x : D_x \to D_x$ induced by the complex structure of $\mathbb{C}^n$. It turns out that the pair $(\mathcal{D}, J)$ automatically satisfies the integrability conditions (4.1) and hence it is always a CR structure.

Assume now that $(\mathcal{D}, J)$ is a CR structure of codimension $r$, with $\mathcal{D}$ regular distribution of type $m$. Then the structure $(\mathcal{D}, J)$ is canonically associated with a Tanaka structure of type $(m, G^0)$, where $G^0$ is the group of automorphisms of $m$, whose restriction to $m^{-1}$ commute with $J$.

N. Tanaka developed a theory of prolongations of Tanaka structures, which is a very deep generalization of the prolongation procedure for the standard $G$-structures.

First of all, he introduced the concept of full prolongation of a Tanaka algebra $m + g^0$

$$(m + g^0)^\infty = m + g^0 + g^1 + \cdots + g^k + \cdots,$$

which is a maximal graded Lie algebra having non-positive part isomorphic to $m + g^0$. A Tanaka algebra $m + g^0$ is called of finite type $k$ if there exists an integer $k$ so that $g^{k-1} \neq 0$ and $g^{k+i} = 0$ for any $i \geq 0$.

After this, Tanaka associated with a given Tanaka structure $\pi : P \to M$ of type $(m, G^0)$ a tower of principal bundles

$$\ldots P^{(k+1)} \xrightarrow{\pi_k} P^{(k)} \to \cdots \to P^{(1)} \xrightarrow{\pi_1} P \xrightarrow{\pi} M,$$

where each $\pi_k : P^{(k)} \to P^{(k-1)}$ is a principal bundle with commutative structure group $G^k$ having as Lie algebra

$$g^k = g^k + g^{k+1} + \cdots / (g^{k+1} + g^{k+2} + \cdots).$$
In general, the bundle $\pi_j : P^{(j)} \to P^{(j-1)}$ is not a $G^j$-structure, but it is always a quotient of some $G$-structure on $P^{(j-1)}$ by some suitable Lie group of transformations. Furthermore, if the Tanaka algebra $m + g^0$ has finite type $k$, then the projection $\pi_k : P^{(k)} \to P^{(k-1)}$ determines an absolute parallelism on $P^{(k-1)}$ and the full system of invariants of this absolute parallelism on $P(M) = P^{(k-1)}$ gives the full system of invariants of the Tanaka structure. In particular, it follows that the automorphism group of such Tanaka structure is a Lie group of dimension less or equal than $\dim P^{(k-1)}$.

We observe that an arbitrary Tanaka algebra is very often of infinite type and that Tanaka structures associated with such Tanaka algebras have infinite dimensional automorphism groups. In particular, the group of automorphisms of a generic CR structure is infinite dimensional; nevertheless, in [26], N. Tanaka found a very nice criterion for a CR structure to have a finite dimensional automorphism group (see also [16] and §6 in this paper).

As for the case of $G$-structures, Tanaka’s prolongation scheme depends on choices of some complementary subspaces (see Theorem 7.6, below). It turns out that, in general, the natural projection $\pi^{k-1} : P^{(k-1)} \to M$ is not a principal bundle over $M$ and the absolute parallelism, determined by the bundle $\pi_k : P^{(k)} \to P^{(k-1)}$, is not a Cartan connection. On the other hand, when the Tanaka algebra $m + g^0$ is equal to the non-positive part of a graded semisimple Lie algebra

$$l = g^{-d} + \cdots + g^{-1} + g^0 + g^1 + \cdots + g^d = m + g^0 + g^1 + \cdots + g^d$$

and if the full prolongation of $m + g^0$ coincides with $(m + g^0)_{\infty} = l$, N. Tanaka in [28] and A. Čap and H. Schichl in [7] gave two new constructions, which are independent from original Tanaka’s prolongation procedure and which associate with a given Tanaka structure $\pi : P \to M$ of type $(m,G^0)$, with $G^0$ connected Lie group associated with the Lie algebra $g^0$, a principal bundle $\pi' : P' \to M$ whose structure group is the parabolic subgroup $Q$ of the adjoint group $L$ of the Lie algebra $l$, having Lie algebra given by the non-negative part of $l$, i.e. $Lie(Q) = g^0 + g^1 + \cdots + g^k \subset l$. Such constructions determine also an absolute parallelism on $P'$, which is a Cartan connection with values in $l$.

We recall that Tanaka structures of type $(m,G^0)$, with $m + g^0$ as above, are called parabolic structures or parabolic geometries and that the flag manifold
\( L/Q \) is the standard model of such parabolic geometries (see e.g. [7], [8], [21], [22], [14]).

We finally want to mention that T. Morimoto presented in [19] a prolongation theory for geometric structures on filtered manifold, which is different from Tanaka’s theory, and obtained the existence of a canonical Cartan connection for parabolic structures as a corollary of a more general result (see [19], Prop. 3.10.1 and remarks in [7], §1).

In this paper, we give a short exposition of Tanaka’s theory of prolongations. We also show that when the Tanaka algebra \( m + g^0 \) is the non-positive part of a semisimple Lie algebra \( l \) and its full prolongation coincides with \( l \), then an appropriate Tanaka prolongation of a parabolic structure \( \pi : P \to M \) of type \( (m, G^0) \) is a principal bundle endowed with a Cartan connection. More precisely, we show that, at each step, it is possible to choose the complementary subspaces in such a way that the natural projection \( \pi_d : P^{(d)} \to M \) of the \( d \)-th prolongation \( P^{(d)} \) onto \( M \) defines a principal bundle structure with structure group \( Q = G^0 \cdot G^1 \cdot \ldots \cdot G^d \) and that the \( \{e\} \)-structure \( \pi^{(d+1)} : P^{(d+1)} \to P^{(d)} \) determines a Cartan connection on \( P^{(d)} \). This gives a new proof of the results of [28] and [7].

2. Fundamental algebra of a distribution

Regular distributions of type \( m \).

Let \( \mathcal{D} \) be a distribution of dimension \( m \) on a manifold \( M^n \) and let us denote by \( \Gamma(\mathcal{D}) \subset \mathfrak{X}(M) \) the space of vector fields \( X \in \mathfrak{X}(M) \), taking values in \( \mathcal{D} \). We also define inductively the following spaces of vector fields for any integer \( i \leq -1 \):

\[
\Gamma(\mathcal{D})_{-i} = \Gamma(\mathcal{D}) , \quad \Gamma(\mathcal{D})_{-i} = \Gamma(\mathcal{D})_{-i+1} + [\Gamma(\mathcal{D}), \Gamma(\mathcal{D})_{-i+1}]
\]

and, for any \( x \in M \), we set \( \mathcal{D}_{-i}(x) \) to be the subspace of \( T_x M \) spanned by the vector fields in \( \Gamma(\mathcal{D})_{-i} \)

\[
\mathcal{D}_{-i}(x) = \{ X|_x : X \in \Gamma(\mathcal{D})_{-i} \} .
\]
For any \( x \in M \), we have that
\[
\cdots = D_{-d-1}(x) = D_{-d}(x) \supseteq D_{-d+1}(x) \supseteq \cdots \supseteq D_{-2}(x) \supseteq D_{-1}(x) = D_x
\]
is a filtration of \( T_xM \). Also, since \([\Gamma(D)_{-i}, \Gamma(D)_{-j}] \subset \Gamma(D)_{-i-j}\), we have that the Lie bracket in \( \mathfrak{X}(M) \) induces a structure of graded Lie algebra on the space
\[
m(x) = m^{-d}(x) + m^{-d+1}(x) + \cdots + m^{-1}(x),
\]
where \( m(x) = \text{gr}(T_xM) \) is the graded vector space of the filtered space \( T_xM \), i.e. \( m^{-1}(x) \overset{\text{def}}{=} D_{-1}(x) = D_x \) and \( m^{-i}(x) \overset{\text{def}}{=} D_{-i}(x)/D_{-i+1}(x) \) for all \( i > 1 \).

The space \( m(x) \) is a negatively graded Lie algebra (i.e. \( m^i(x) = 0 \) for \( i \geq 0 \)) generated by \( m^{-1}(x) \). A finite dimensional graded Lie algebra with these two properties is called fundamental. The graded Lie algebra \( m(x) \) is called fundamental algebra of \( D \) at the point \( x \).

**Definition 2.1** Let \( m = m^{-d} + m^{-d+1} + \cdots + m^{-1} \) be a fundamental Lie algebra. A distribution \( \mathcal{D} \) on a manifold \( M \) is called regular of type \( m \) if

i) for any \( x \in M \), the fundamental algebra \( m(x) \) is isomorphic to \( m \)

ii) \( \dim M = \dim m \).

The integer \( d \) is called depth of \( \mathcal{D} \).

Note that if \( \mathcal{D} \) is a regular distribution, then the subspaces \( \mathcal{D}_{-i}(x), x \in M, -d \leq i \leq -1 \), determine \( d \) distributions \( \mathcal{D}_i \) and such distributions constitute a flag of distributions, that is a nested sequence
\[
TM = \mathcal{D}_{-d} \supset \mathcal{D}_{-d+1} \supset \mathcal{D}_{-d+2} \supset \cdots
\]
such that \([\Gamma(\mathcal{D})_i, \Gamma(\mathcal{D})_j] \subset \Gamma(\mathcal{D})_{i+j}\).
3. Tanaka algebras $\mathfrak{m} + \mathfrak{g}^0$ and Tanaka structures

Let $\mathfrak{m} = \mathfrak{m}^{-d} + \cdots + \mathfrak{m}^{-1}$ be a fundamental algebra of depth $d$. We denote by $\text{Aut}(\mathfrak{m})$ the group of all (grading preserving) automorphisms of the Lie algebra $\mathfrak{m}$ and by $\text{aut}(\mathfrak{m})$ the Lie algebra of $\text{Aut}(\mathfrak{m})$. Note that, since $\mathfrak{m}$ is generated by $\mathfrak{m}^{-1}$, the restriction map $\text{Aut}(\mathfrak{m}) \to \text{Aut}(\mathfrak{m})|_{\mathfrak{m}^{-1}}$ is an exact representation.

**Definition 3.1** Let $\mathfrak{m}$ be a fundamental algebra of depth $d$ and $G^0 \subset \text{Aut}(\mathfrak{m})$ a subgroup of its automorphism group. The pair $(\mathfrak{m}, G^0)$ is called a **Tanaka pair of depth** $d$.

If $G^0$ is connected, then the Tanaka pair is canonically associated with the graded Lie algebra (called **Tanaka algebra**) $\mathfrak{m} + \mathfrak{g}^0 = \mathfrak{m}^{-d} + \cdots + \mathfrak{m}^{-1} + \mathfrak{g}^0$, $\mathfrak{g}^0 = \text{Lie}(G^0)$.

Let $\mathcal{D} \subset TM$ be a regular distribution of type $\mathfrak{m}$. We call an isomorphism $\xi : \mathcal{D}_x \to \mathfrak{m}^{-1}$ an **$\mathfrak{m}$-adapted coframe** on $\mathcal{D}_x$ if it can be extended to an isomorphism $\hat{\xi} : \mathfrak{m}(x) \to \mathfrak{m}$ of graded Lie algebras.

We remark that if such an extension $\hat{\xi} : \mathfrak{m}(x) \to \mathfrak{m}$ exists, then it is unique. In fact, any two such extended isomorphisms have to be related by an automorphism $\alpha$ of $\mathfrak{m}$, which restricts to the identity map on $\mathfrak{m}^{-1}(x)$, and this implies that $\alpha = \text{Id}$.

**Definition 3.2** Let $(\mathfrak{m}, G^0)$ be a Tanaka pair of depth $d$ and $\mathcal{D}$ a regular distribution of type $\mathfrak{m}$ on a manifold $M$. A **partial $G^0$-structure** or **Tanaka structure of depth** $d$ is a principal $G^0$-bundle $\pi : P \to M$ of $\mathfrak{m}$-adapted coframes on $\mathcal{D}_x \subset T_x M$, $x \in M$.

Note that the only regular distribution of depth 1 is the trivial distribution $\mathcal{D} = TM$ and that its type $\mathfrak{m} = \mathfrak{m}^{-1}$ is the commutative Lie algebra of dimension $n = \dim M$. Therefore a Tanaka structure of depth 1 is simply a $G^0$-structure on $M$, that is a principal subbundle of the bundle of coframes with structure group $G^0 \subset \text{GL}_n(\mathbb{R})$ (see §5 below).
4. Tanaka algebras and Tanaka structures of CR type
- Regular CR structures

In this section, we introduce the notion of regular CR structures and we show that they are special cases of Tanaka structures (for general reference on CR structures, see e.g. [2], [29]).

A CR structure of codimension \( r \) on a manifold \( M \) is a pair \((\mathcal{D}, J)\) formed by a distribution \( \mathcal{D} \subset TM \) of codimension \( r \) and a smooth family of complex structures \( J_x : \mathcal{D}_x \to \mathcal{D}_x \) which satisfies the following integrability condition: the \( J \)-eigenspace distribution \( \mathcal{D}^{10} \subset T \mathbb{C} \mathcal{N} \), corresponding to the eigenvalue \( +i \), is involutive, that is

\[
[D^{10}, D^{10}] \subset D^{10}.
\]

This integrability condition can be also stated saying that, for any two vector fields \( X, Y \) in \( D \),

\[
[JX, Y] + [X, JY] \in D,
\]

\[
[JX, Y] - [X, Y] - J([JX, Y] + [X, JY]) = 0
\]

A fundamental algebra \( \mathfrak{m} \) together with a complex structure \( J \) on \( \mathfrak{m}^{-1} \) is called CR fundamental algebra if for any \( X, Y \in \mathfrak{m}^{-1} \)

\[
[JX, JY] = [X, Y].
\]

In this case, we will always denote by \( \text{Aut}(\mathfrak{m}, J) \) the subgroup of \( \text{Aut}(\mathfrak{m}) \), whose restriction on \( \mathfrak{m}^{-1} \) commutes with \( J \). It is isomorphic to a subgroup of \( \text{GL}(\mathfrak{m}^{-1}, J) = \text{GL}_{m-r}(\mathbb{C}) \).

Observe that if \((M, \mathcal{D}, J)\) is a CR manifold, then by (4.2), for any \( x \in M \) the pair \((\mathfrak{m}(x), J_x)\) is a CR fundamental algebra.

Now, given a CR fundamental algebra \((\mathfrak{m}, J)\), a CR structure \((\mathcal{D}, J)\) on \( M \) is called regular CR structure of type \((\mathfrak{m}, J)\), if \((\mathfrak{m}(x), J_x)\) is isomorphic to \((\mathfrak{m}, J)\) for any \( x \in M \). In this case, the distribution \( \mathcal{D} \) is clearly regular of type \( \mathfrak{m} \) and we will call depth of the CR structure the depth of \( \mathfrak{m} \).
Proposition 4.1 [26] Let $\mathcal{D}$ be a regular distribution of type $m$ on a manifold $M$. Then any regular CR structures $(\mathcal{D}, J)$ of type $(m, J)$ naturally defines a Tanaka structure $\pi : P \to M$ of type $(m, \text{Aut}(m, J))$, which consists of all $m$-adapted coframes $\xi : D_x = m^{-1}(x) \to m^{-1}$, $x \in M$, such that $J \circ \xi = \xi \circ J_x$.

Conversely, let $(m, J)$ be a CR fundamental algebra and $\pi : P \to M$ a Tanaka structure of type $(m, \text{Aut}(m, J))$. Then $P$ defines a family of complex structures $J_x : D_x \to D_x$ on the distribution $\mathcal{D}$ and the pair $(\mathcal{D}, J)$ is a CR structure if and only it satisfies the integrability condition (4.3).

For a given CR fundamental algebra $(m, J)$, the Tanaka algebra

$$m + \text{Lie}(\text{Aut}(m, J))$$

will be called the CR Tanaka algebra associated with $(m, J)$. The Tanaka structure $\pi : P \to M$ described in Proposition 4.1 will be called the CR Tanaka structure associated with the regular CR structure $(\mathcal{D}, J)$.

Consider now a CR fundamental algebra $(m = m^{-2} + m^{-1}, J)$ of depth 2. In this case the value $r = \dim m^{-2}$ coincides with the codimension of the regular CR structures of type $(m, J)$. Furthermore, the Lie algebra $m$ is metabelian with center $m^{-2}$ and the Lie bracket is completely determined by a linear map $m^{-1} \wedge m^{-1} \to m^{-2}$. Consider a basis $e_1, \ldots, e_r$ for $m^{-2}$. Then the only non-trivial brackets in $m = m^{-2} + m^{-1}$ are given by

$$[X, Y] = \sum_{i=1}^{r} h^i(X, Y) e_i, \quad X, Y \in m^{-1},$$

where the $h^i$ are linearly independent Hermitian skew-symmetric forms. Conversely, any $r$-tuple of linearly independent Hermitian skew-symmetric forms on $m^{-1}$ defines a CR fundamental algebra $m = m^{-2} + m^{-1}$.

Assume now that $r = \dim m^{-2} = 1$. Then any CR fundamental algebra $(m = m^{-2} + m^{-1}, J)$ is given by just one non-zero Hermitian skew-symmetric form $h$ on the complex space $(m^{-1}, J)$, determined up to scaling. The Hermitian symmetric form $H = h(J\cdot, \cdot)$ is called Levi form and its signature is the only invariant of the CR fundamental algebra $m$. 


5. G-structures, i.e. Tanaka structures of depth one

In this section we recall some basic facts on the theory of prolongation of G-structures, i.e. Tanaka structures of depth one.

5.1 - Definition of G-structures.

Let $G \subset GL(V)$, $V = \mathbb{R}^n$, be a linear group with Lie algebra $\mathfrak{g}$. A G-structure on a $n$-dimensional manifold $M$ is a principal $G$-subbundle of the $GL(V)$-bundle of coframes on $M$. We recall that a coframe at a point $x \in M$ is an isomorphism $u : T_xM \to V$.

A G-structure can be also defined in a more intrinsic way as follows.

**Definition 5.1** A G-structure on a manifold $M$ is a principal $G$-bundle $\pi : P \to M$ equipped with a $V$-valued $G$-equivariant strictly horizontal 1-form $\theta : TP \to V$. We recall that a 1-form $\theta$ is called horizontal if $\text{Ker} \theta = T^\text{Vert} P$, where $T^\text{Vert} P$ is the vertical subbundle of the tangent bundle $TP$, and that a $V$-valued 1-form $\theta$ is called $G$-equivariant if $(g^* \theta)_u(X) \overset{\text{def}}{=} \theta_{g(u)}(g_*X) = g \circ \theta(X)$ for any $g \in G$ and $X \in V$.

The form $\theta$ is called the soldering form of the G-structure $\pi$.

Using the form $\theta$, we may associate to any point $u \in P$ a coframe $\hat{u}$, i.e. an isomorphism

$$\hat{u} : T_{\pi(u)}M \to V,$$

which is determined by the relation

$$\hat{u} \circ \pi_* = \theta_u.$$

The correspondence $u \mapsto \hat{u}$ determines an identification of $P$ with a principal $G$-subbundle of the coframe bundle $\mathcal{C}F(M)$. 

5.2 - Full prolongation of a linear Lie algebra.

Let \( g^0 \subset gl(V) \), \( V = \mathbb{R}^n \), be a linear Lie algebra. The first prolongation of \( g^0 \) is the subspace of \( g^0 \otimes V^* = \text{Hom}(V, g^0) \) defined as

\[ g^1 = g^0 \otimes V^* \cap V \otimes S^2(V^*) . \]

Note that \( g^1 \) has a natural embedding as abelian (nilpotent) Lie subalgebra of \( gl(V + g^0) \) acting trivially on \( g^0 \).

We define inductively the \( k \)-th prolongation of \( g^0 \) as the subspace of \( g^{k-1} \otimes V^* = \text{Hom}(V, g^{k-1}) \) given by

\[ g^k = g^{k-1} \otimes V^* \cap V \otimes S^{k+1}(V^*) . \]

Again, \( g^k \) has a natural embedding as an abelian (nilpotent) Lie subalgebra \( gl(V + g^0 + \cdots + g^{k-1}) \), which acts trivially on \( g^0 + \cdots + g^{k-1} \).

We also denote by \( G^k = \{ A = Id + B \mid B \in g^k \} \) the unipotent subgroup of \( GL(V + g^0 + \cdots + g^{k-1}) \) with Lie algebra \( g^k \).

We set \( g^{-1} = V \). Since each \( g^k \), \( k \geq -1 \), is a subspace of \( V \otimes S^{k+1}V^* \), then the direct sum

\[ (V + g^0)_{\infty} = V + g^0 + g^1 + \cdots \]

is a graded subspace of the graded Lie algebra \( V \otimes S(V^*) = \sum_{k=-1}^\infty V \otimes S^{k+1}V^* \) of formal vector fields on \( V \). It is easy to check that \( (V + g^0)_{\infty} \) is a graded subalgebra of \( V \otimes S(V^*) \). It is called the full prolongation of the linear algebra \( g^0 \).

A linear algebra \( g^0 \) is called of finite type if its full prolongation is finite dimensional with \( g^{k-1} \neq \{0\} \) and with \( g^\ell = \{0\} \) for any \( \ell \geq k \).

A useful criterium for \( g^0 \) to be of finite type is given in the following theorem.

**Theorem 5.2** [10] Let \( g^0 \subset gl(V) \), \( V = \mathbb{C}^n \), be a complex linear Lie algebra. Then it is of finite type if and only if it contains no element of rank one.
Moreover, for a completely reducible linear Lie algebra over \( \mathbb{R} \) or \( \mathbb{C} \), we have the following more precise result.

**Theorem 5.3** [15], [20] Let \( g^0 \subset gl(V) \) be a completely reducible linear Lie algebra. If its second prolongation \( g^2 \neq \{0\} \), then \( g^0 \) is of infinite type.

Moreover, if \( g^0 \) is irreducible and of finite type with \( g^1 \neq \{0\} \), then the full prolongation \( g = g^{-1} + g^0 + g^1 \) is a simple graded Lie algebra.

The complete list of all such Lie algebra is given in [20].

5.3 - Derived \( G' \)-structures and prolongations of \( G^0 \)-structure.

Denote by \( i_u : T^\text{Vert}_uP \to g^0, u \in P \), the canonical isomorphism of the vertical subspace \( T^\text{Vert}_uP \subset T_uP \) onto the Lie algebra \( g^0 \) of \( G^0 \), whose inverse map is

\[
g^0 \ni A \mapsto A_u = \left. \frac{d\exp(tA) \circ u}{dt} \right|_{t=0} \in T^\text{Vert}_uP.
\]

An extension of the map \( i : T^\text{Vert}_uP \to g^0 \) to a \( G^0 \)-equivariant \( g^0 \)-valued 1-form \( \omega : TP \to g \) on \( P \) is called connection form and it defines a connection in the \( G^0 \)-bundle \( \pi \). Such extension \( \omega \) is uniquely defined by the kernel \( \text{Ker} \omega \), which is an arbitrary \( G^0 \)-invariant field \( u \mapsto \mathcal{H}_u = \text{Ker} \omega_u \) of horizontal subspaces in \( TP \), that is of subspaces such that

\[
\forall u \in P, \quad T_uP = T_u^\text{Vert}P \oplus \mathcal{H}_u.
\]

We remark that a connection \( \omega \) in \( \pi : P \to M \) naturally determines a linear connection on \( M \) with an operator of covariant derivative \( \nabla \) (see e.g. [23]). Notice also that the restriction \( \theta_{\mathcal{H}_u} \overset{\text{def}}{=} \theta_{|\mathcal{H}_u} \) to a horizontal subspace \( \mathcal{H}_u \subset T_uP \), determines an isomorphism

\[
\theta_{\mathcal{H}_u} : \mathcal{H}_u \xrightarrow{\cong} V.
\]

Hence we can associate with \( \mathcal{H}_u \) an isomorphism

\[
\hat{\mathcal{H}}_u = \theta_{\mathcal{H}_u} \oplus i_u : T_uP = \mathcal{H}_u + T_u^\text{Vert}P \xrightarrow{\cong} V' \overset{\text{def}}{=} V + g^0,
\]

that is a coframe at \( u \in P \).
Let us denote by $G' = \text{Hom}(V, g^0) \subset GL(V')$ the vector subgroup of $GL(V')$, acting linearly on the vector space $V' = V + g^0$ as follows:

$$G' \ni B \cdot : (v + A) \mapsto v + (A + B_v), \quad v \in V, \ A \in g^0.$$

Let also $P^1$ be the manifold of all horizontal subspaces in $TP$, or, equivalently, the manifold of 1-jets of sections of the bundle $\pi: P \to M$. Then the group $G'$ acts naturally and freely on $P^1$ (on the left) with orbit space $P^1/G' = P$. Moreover, the natural projection $\pi_1: P^1 \to P$ determines a principal $G'$-bundle structure on $P^1$. Using Definition 5.1, this bundle becomes a $G'$-structure if we consider as soldering form the 1-form $\theta_1: TP^1 \to V' = V + g^0$ defined by

$$(5.1) \quad \theta_1^H = \hat{\mathcal{H}} \circ \pi_{1*} = (\theta_{\mathcal{H}} \oplus \iota_u) \circ \pi_{1*}, \quad u = \pi^1(\mathcal{H}),$$

for any $\mathcal{H} \in P^1$.

These observations are summarized in the following proposition.

**Proposition 5.4** Let $\pi: P \to M$ be a $G^0$-structure with soldering form $\theta: TP \to V$. Then the bundle $\pi_1: P^1 \to P$ of 1-jets of sections of $\pi: P \to M$, is a $G'$-structure with soldering form $\theta^1: TP^1 \to V'$ defined by (5.1), where $G' = \text{Hom}(V, g^0) \subset GL(V')$.

The form $\theta^1: TP^1 \to V'$ is $G^0$-equivariant with respect to the natural action of $G^0$ on the manifold $P^1$ and on the vector space $V' = V + g^0$.

A connection in the $G^0$-structure $\pi: P \to M$ may be identified with a $G^0$-equivariant section

$$\hat{\mathcal{H}}: P \ni u \mapsto \mathcal{H}_u \in P^1$$

of the bundle $\pi_1: P^1 \to P$.

For any $G^0$-structure $\pi: P \to M$, there exists a canonical $V \otimes \Lambda^2 V^*$-valued torsion function on the manifold $P^1$ of horizontal subspaces

$$\tau: P^1 \to V \otimes \Lambda^2 V^*,$$

defined as follows. For any $\mathcal{H} \in P^1$ we set

$$\tau_\mathcal{H}(X, Y) = d\theta_u(\theta_\mathcal{H}^{-1}(X), \theta_\mathcal{H}^{-1}(Y)) \quad X, Y \in V,$$
where $\theta_H \overset{\text{def}}{=} \theta_u|_H, u = \pi_1(H) \in P$.

The torsion function $\tau$ is equivariant with respect to the natural action of the semidirect product $G^0 \cdot G'$, where the action of $G'$ on the vector space $V \otimes \Lambda^2 V^*$ is given by

$$G' \ni B : V \otimes \Lambda^2 V^* \ni T \to T + \partial B.$$ 

Here $\partial$ denotes the map

$$\partial : G' = \text{Hom}(V, g^0) = g^0 \otimes V^* \subset \text{Hom}(V \otimes V, V) \longrightarrow V \otimes \Lambda^2 V^*$$

given by the Spencer operator of alternation, which maps $\text{Hom}(V \otimes V, V)$ into $\text{Hom}(\Lambda^2 V, V)$.

Observe that if $\sigma_\omega : P \to P^1$ is a section of the bundle $\pi_1 : P^1 \to P$ which defines a connection $\omega$, then for any $u \in P$, the torsion $\tau_{H_u} = \sigma_\omega(u)$, is the torsion tensor of the connection $\omega$ computed with respect to the coframe $u$. The function $\tau \circ \sigma_\omega : P \to V \otimes \Lambda^2 V^*$ is called the torsion function of the connection $\omega$.

Let us now fix a complement $C$ to $\partial(g \otimes V^*)$ in $V \otimes \Lambda^2 V^*$:

$$V \otimes \Lambda^2 V^* = \partial(g^0 \otimes V^*) \oplus C.$$ 

Then we set

$$P^{(1)} = \tau^{-1}(C) = \{ H \in P^1 : \tau_H \in C \}.$$ 

The induced projection $\pi_1 : P^{(1)} \to P$ determines a structure of principal bundle with commutative structure group

$$G^1 = (g^0 \otimes V^*) \cap V \otimes S^2 V^* \subset G' = g^0 \otimes V^* \subset GL(V).$$

The bundle $\pi_1 : P^{(1)} \to P$ equipped with the $V'$-valued 1-form $\theta^{(1)} = \theta^{(1)}|_{P^{(1)}}$ is a $G^{(1)}$-structure. It is called the first prolongation of the $G$-structure $\pi : P \to M$.

In general, the group $G^0$ does not act on the first prolongation $P^{(1)}$. However, if the complementary subspace $C$ to $\partial(g^0 \otimes V^*)$ in $V \otimes \Lambda^2 V^*$ is $G^0$-invariant, then the natural action of $G^0$ on bundle of coframes $\mathcal{C}F(P)$ of $P$ preserves the
subbundle $P^{(1)} \subset \mathcal{CF}(P)$. In fact, for any horizontal subspace $\mathcal{H} \subset P^1$ and any $g \in G^0$, we have that
\[
\tau_{g, \mathcal{H}}(X, Y) = d\theta_g \cdot u(\theta_g^{-1}(g^{-1}X), \theta_g^{-1}(g^{-1}Y)) = g \circ \tau_{\mathcal{H}}(g^{-1}X, g^{-1}Y), \quad X, Y \in V
\]
and hence, since $\mathcal{C}$ is $G^0$-invariant, we have that $\tau_{\mathcal{H}} \in \tau^{-1}(\mathcal{C}) = P^{(1)}$ if and only if also $\tau_{g, \mathcal{H}} \in \tau^{-1}(\mathcal{C}) = P^{(1)}$.

This action of $G^0$ together with the action of $G^1$ on $P^{(1)}$ determines an action of the group $G^0 \cdot G^1 \subset GL(V + g^0)$, corresponding to the Lie algebra $g^0 + g^1$, and the quotient $P^{(1)}/G^0 \cdot G^1$ is equal to $M$. In other words, in this case, $\pi \circ \pi_1 : P^{(1)} \to M$ is a principal bundle with structure group $G^0 \cdot G^1$.

Iterating the process for constructing the first prolongation, we get a tower of $G$-structures
\[
\cdots \to P^{(k+1)} \xrightarrow{\pi_{k+1}} P^{(k)} \xrightarrow{\pi_k} \cdots \xrightarrow{\pi_2} P^{(1)} \xrightarrow{\pi_1} P^{(0)} \xrightarrow{\pi} M
\]
where, for any $k \geq 1$, $\pi_k : P^{(k)} \to P^{(k-1)}$ is $G$-structure with structure group $G = G^k \subset GL(V + g^0 + \cdots + g^{k-1})$.

Moreover, since the construction is natural, any (local) automorphism $\phi : M \to M$ of the $G^0$-structure $\pi : P \to M$ induces a (local) automorphism of the $G^k$-structure $\pi_k : P^{(k)} \to P^{(k-1)}$ for any $k$ and vice versa. In particular the automorphism group $\text{Aut}(\pi)$ of the $G^0$-structure $\pi : P \to M$ is isomorphic with the automorphism group $\text{Aut}(\pi_k)$ of the $G^k$-structure $\pi_k : P^{(k)} \to P^{(k-1)}$, for any $k \geq 1$.

Assume now that $G^0$ is of finite type $k$. Then $G^k = \{e = Id\}$ and $\pi_k : P^{(k)} \to P^{(k-1)}$ is an $\{e\}$-structure, i.e. a field of frames on $P^{(k-1)}$. By Kobayashi’s theorem (see [11]), we get the following result.

**Theorem 5.5** Let $\pi : P \to M$ be a $G^0$-structure, with $g^0 = \text{Lie}(G^0)$ of finite $k$. Then, the automorphism group $\text{Aut}(\pi)$ ($\cong \text{Aut}(\pi_k)$) is a Lie group of dimension less or equal to
\[
\dim P^{(k)} = \dim(V + g^0 + \cdots + g^{k-1})
\]