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Editors
Maria Stella Mongiovì
Michele Sciacca
Salvatore Triolo

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www.aracneeditrice.it
info@aracneeditrice.it

via Quarto Negroni, 15
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Contents

Geometry of Rings: An Elementary Introduction

Gioia Failla[1] Mustapha Lahyane[2] Rosanna Utano[3]
Juan Bosco Frías Medina[2]

[1]DIIES, University of Reggio Calabria, Via Graziella, Feo di Vito, Reggio Calabria. Italy
[2]Instituto de Física y Matemáticas (IFM), Universidad Michoacana de San Nicolás de Hidalgo (UMSNH). Edificio C-3, Ciudad Universitaria. C. P. 58040, Morelia, Michoacán. Mexico
[3]Department of Mathematical and Computer Sciences, Physical and Earth Sciences, University of Messina, Viale Ferdinando Stagno D' Alcontres 31, 98166 Messina. Italy

E-mail(s): gioia.failla@unirc.it; lahyane@ifm.umich.mx, utanor@unime.it, boscof@ifm.umich.mx

Abstract

The aim of this work is to give an elementary introduction to the geometry of any given ring. In particular, it is self-contained, and offers not only a generalization of classical algebraic, differential, and analytic geometries but also an efficient way to be familiar, without much efforts, to one of the basic and necessary tools used frequently in Algebraic Geometry.

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Contents

1 Introduction

Given a ring R, one may naturally associate to it a geometric object which consists of a pair $(\mathrm{Spec}(R), \mathcal{O}_{\mathrm{Spec}(R)})$ where Spec(R) is the set of all prime ideals of R and $\mathcal{O}_{\mathrm{Spec}(R)}$ is a sheaf of rings on Spec(R). Here, a ring means a commutative ring with a unit, and it is assumed that such ring has at least two elements, except in cases where a ring may be reduced to only one element, and when these cases occur, we mention them explicitly. Such geometric object is called the *Spectrum* of the ring, and is a particular case of what is generally known as an *affine scheme*.

At the end of the 50's, Alexander Grothendieck and Jean-Alexandre Dieudonné formulated the Scheme Theory which is the base for the modern Algebraic Geometry. In recent years, it has been proved the power of such theory not only for solving problems coming from Algebraic Geometry, but also for their use in other areas such as Differential and Complex Geometry, Number Theory, Differential Equations, etc. Furthermore, there exist concrete applications of such theory to other areas such as Coding Theory, Cryptography, Mathematical Physics, Phylogenetics and Robotics to mention a few.

The Scheme Theory is based on the language of Commutative Algebra, and the tools coming from Sheaf Theory. However, it could be difficult for the beginners to enter in this area, because the classic texts of study Algebraic Geometry have a level that could be not adequate. The aim of this survey is to present in a self-contained, detailed and elementary introduction to Sheaf Theory, and also to introduce in the same way, the heart notion of an affine scheme, which is one of the most basic objects in Scheme Theory, and some of their fundamental properties. The contents of this survey is based on [2] regarding the localization of modules and rings, and on [1] and [3] regarding Sheaf Theory, and the geometric object associated to a ring.

In Section 2, we give a detailed presentation of Sheaf Theory. In the first subsubsection, we introduce the notion of a sheaf and morphism of sheaves, and also we study some of their properties. as well as. their local data. The subsubsection two is devoted to the construction of some important sheaves. The notion of short exact sequence of sheaves is treated in subsubsection three while in subsubsection four, we handle the sheaves of modules. Finally, Section 3 deal with the geometry of rings: after a detailed study of the localization of modules and rings at multiplicative sets, we start to define the topological space associated to any given ring and to give some properties of such topological space. Next, we define a canonical sheaf of rings on the topological space just obtained, and study in a detailed way its properties. In particular, we define the geometric object associated to any given ring as the spectrum of such ring. Then, we study morphisms between spectra, and show that these morphisms have a purely algebraic character. At the end, we offer some important properties regarding the basic open subsets of a given spectrum.

2 Sheaf Theory

In this section, we present an introduction to the theory of sheaves with full details. We only assume the reader to be familiar with some fundamental, but elementary, notions of *Commutative Algebra*, *General Topology*, and *Modern Algebra*. This section is based also on the first chapter of the unpublished book [1].

2.1 Presheaves and their Fundamental Related Notions

We review in this subsection some basic concepts, and notions from *Sheaf Theory* that we need in order to understand the geometric object associated to a given ring. So, we will first deal with presheaves, sheaves, and stalks, as well as, some of their standard properties.

2.1.1 Presheaves and Sheaves

Definition 2.1. Let X be a topological space. A *presheaf of abelian groups* on X is a pair $(\mathcal{F}, \rho_{\mathcal{F}})$, where $\mathcal{F}$ assigns an abelian group $\mathcal{F}(U)$ to every open subset U of X, and $\rho_{\mathcal{F}}$ assigns a homomorphism of abelian groups ${\rho_{\mathcal{F}}}_V^U : \mathcal{F}(U) \to \mathcal{F}(V)$ to every open subsets U and V of X with $V \subseteq U$, furthermore these data should satisfy the following requirements:

PS1. $\mathcal{F}(\emptyset) = \{0\}$,

PS2. ${\rho_{\mathcal{F}}}_U^U$ is the identity map for every open subset U of X, and

PS3. ${\rho_{\mathcal{F}}}_W^V \circ {\rho_{\mathcal{F}}}_V^U = {\rho_{\mathcal{F}}}_W^U$ for every open subsets U, V and W of X with $W \subseteq V \subseteq U$.

Definition 2.2. Let X be a topological space. A *sheaf of abelian groups* on X is a presheaf $(\mathcal{F}, \rho_{\mathcal{F}})$ of abelian groups on X such that for every open subset U of X and every open covering $\big(U_i\big)_{i\in I}$ of U, the following requirements hold:

S1. Let f be an element of $\mathcal{F}(U)$. If ${\rho_{\mathcal{F}}}_{U_i}^U(f) = 0_{\mathcal{F}(U_i)}$ for all $i \in I$, then $f = 0_{\mathcal{F}(U)}$.

S2. If $f_i \in \mathcal{F}(U_i)$ for every $i \in I$ such that ${\rho_{\mathcal{F}}}_{U_i\cap U_j}^{U_i}(f_i) = {\rho_{\mathcal{F}}}_{U_i\cap U_j}^{U_j}(f_j)$ for every $i, j \in I$, then there exists $f \in \mathcal{F}(U)$ satisfying ${\rho_{\mathcal{F}}}_{U_i}^U(f) = f_i$ for all $i \in I$.

Here, we fix some notation, and terminologies: Let $(\mathcal{F}, \rho_{\mathcal{F}})$ be a presheaf of abelian groups on a topological space X.

1. We simply write "$\mathcal{F}$ is a presheaf on X" instead of writing "$(\mathcal{F}, \rho_{\mathcal{F}})$ is a presheaf of abelian groups on X",

2. Let U be an open subset of X. Any element of $\mathcal{F}(U)$ is called a *section of $\mathcal{F}$ over U*. We refer to a section of $\mathcal{F}$ over X as a *global section of $\mathcal{F}$*.

3. Let U and V be open subsets of X such that $V \subseteq U$. The group homomorphism ${\rho_{\mathcal{F}}}_V^U$ is called the *restriction of $\mathcal{F}$ from U to V*, or only *restriction* whenever there is no need to emphasize on $\mathcal{F}$, U and V.

4. Let U and V be open subsets of X such that $V \subseteq U$, and let h be a section of $\mathcal{F}$ over U. $h|_V$ means ${\rho_{\mathcal{F}}}_V^U(h)$.

Remark 2.3. Let $\mathcal{F}$ be a presheaf on a topological space X. One may be interested in considering $\mathcal{F}(U)$ as a ring (respectively, a module, etc.) for every open subset U of X, and the restrictions to be homomorphisms of rings (respectively, modules, etc). In such case, we mention that we have a presheaf of rings, (respectively, a presheaf of modules, etc). For example, we will handle the presheaves, and sheaves of modules in Subsection 2.4.

Now, we consider the local data that one can infer immediately from a presheaf $\mathcal{F}$ on a given topological space X. To do so, let p be an element of X. We firstly define the following set:

$$\Gamma_p = \left\{(U, s) \,\middle|\, U \text{ is an open subset of } X \text{ containing } p \text{ and } s \text{ is a section of } \mathcal{F} \text{ over } U\right\}.$$

Next, we define a relation $\sim$ on Γ_p as follows: Let (U, s) and (V, t) be elements of Γ_p. $(U, s) \sim (V, t)$ if there exists an open subset W of X containing p such that $W \subseteq U \cap V$ and $s|_W = t|_W$.
$\sim$ is an equivalence relation. Indeed, let U, V and Z be open subsets of X containing p, and let $s \in \mathcal{F}(U)$, $t \in \mathcal{F}(V)$ and $r \in \mathcal{F}(Z)$:

1. *Reflexivity:* $(U, s) \sim (U, s)$ follows immediately by taking $W = U$,

2. *Symmetry:* If $(U, s) \sim (V, t)$, then there exists an open subset W of X containing p such that $W \subseteq U \cap V$ and $s|_W = t|_W$. Hence, W ensures also $(V, t) \sim (U, s)$, and

3. *Transitivity:* If $(U, s) \sim (V, t)$, and $(V, t) \sim (Z, r)$, then there exist open subsets W_1 and W_2 of X both containing p such that $W_1 \subseteq U \cap V$, $W_2 \subseteq V \cap Z$, $s|_{W_1} = t|_{W_1}$ and $t|_{W_2} = r|_{W_2}$. Therefore, $W_1 \cap W_2$ implies that $(U, s) \sim (Z, r)$. To see this, $W_1 \cap W_2$ is an open subset of X containing p, and is contained in $U \cap Z$. Moreover,

$$s|_{W_1 \cap W_2}$$

$$= (s|_{W_1})|_{W_1\cap W_2}$$

$$= (t|_{W_1})|_{W_1\cap W_2}$$

$$= t|_{W_1\cap W_2}$$

$$= (t|_{W_2})|_{W_1\cap W_2}$$

$$= (r|_{W_2})|_{W_1\cap W_2}$$

$$= r|_{W_1\cap W_2}.$$

So, we are done.

So, we are able to construct the quotient set $\dfrac{\Gamma_p}{\sim}$ of Γ_p by $\sim$ that we denote hereafter by $\mathcal{F}_p$. Let (U, s) be an element of Γ_p, we denote its class in $\mathcal{F}_p$ by $[(U, s)]$. It seems that $\mathcal{F}_p$ inherits an algebraic structure of an abelian group. For, we consider the following operation $+$:

$$\begin{array}{cccc} +: & \mathcal{F}_p\times\mathcal{F}_p & \to & \mathcal{F}_p \\ & \big([(U,s)],[(V,t)]\big) & \mapsto & [(U\cap V, s|_{U\cap V}+t|_{U\cap V})]. \end{array}$$

We need to prove that this operation is binary, that is, $+$ is well-defined: To do so, let $[(U, s)]$, $[(\tilde{U}, \tilde{s})]$, $[(V, t)]$, $[(\tilde{V}, \tilde{t})]$ be elements of $\mathcal{F}_p$ such that $[(U, s)] = [(\tilde{U}, \tilde{s})]$ and $[(V, t)] = [(\tilde{V}, \tilde{t})]$. We have to check the equality $[(U, s)] + [(V, t)] = [(\tilde{U}, \tilde{s})] + [(\tilde{V}, \tilde{t})]$, i.e., $[(U\cap V, s|_{U\cap V}+t|_{U\cap V})] = [(\tilde{U}\cap\tilde{V}, \tilde{s}|_{\tilde{U}\cap\tilde{V}}+\tilde{t}|_{\tilde{U}\cap\tilde{V}})]$. For, $[(U, s)] = [(\tilde{U}, \tilde{s})]$ (respectively, $[(V, t)] = [(\tilde{V}, \tilde{t})]$) implies the existence of an open subset W_1 (respectively,W_2) of X containing p such that $W_1 \subseteq U\cap\tilde{U}$ and $s|_{W_1} = \tilde{s}|_{W_1}$ (respectively, $W_2 \subseteq V\cap\tilde{V}$ and $t|_{W_2} = \tilde{t}|_{W_2}$). Consider the open subset $W_3 = W_1 \cap W_2$ of X which certainely contains p and $W_3 \subseteq (U\cap V)\cap(\tilde{U}\cap\tilde{V})$. Furthermore, the fact that $W_3 \subseteq W_1$ (respectively, $W_3 \subseteq W_2$) ensures $s|_{W_3} = \tilde{s}|_{W_3}$ (respectively, $t|_{W_3} = \tilde{t}|_{W_3}$). So, the equality $s|_{W_3} + t|_{W_3} = \tilde{s}|_{W_3} + \tilde{t}|_{W_3}$ holds, and we are done.
Next, we show that the pair $(\mathcal{F}_p, +)$ is an abelian group. Indeed, let U, V and Z be open subsets of X containing p, and let $s \in \mathcal{F}(U)$, $t \in \mathcal{F}(V)$ and $r \in \mathcal{F}(Z)$.
• *Associativity:*

$$\begin{aligned} &\big([(U,s)]+[(V,t)]\big)+[(Z,r)] \\ &= [(U\cap V, s|_{U\cap V}+t|_{U\cap V})]+[(Z,r)] \\ &= [((U\cap V)\cap Z, (s|_{(U\cap V)\cap Z}+t|_{(U\cap V)\cap Z})+r|_{(U\cap V)\cap Z})] \\ &= [(U\cap(V\cap Z), s|_{U\cap(V\cap Z)}+(t|_{U\cap(V\cap Z)}+r|_{U\cap(V\cap Z)}))] \\ &= [(U,s)]+\big([(V\cap Z, t|_{V\cap Z}+r|_{V\cap Z})]\big) \\ &= [(U,s)]+\big([(V,t)]+[(Z,r)]\big). \end{aligned}$$

- *Commutativity:*

$$\begin{aligned}&[(U,s)]+[(V,t)]\\&=[(U\cap V, s|_{U\cap V}+t|_{U\cap V})]\\&=[(V\cap U, t|_{V\cap U}+s|_{V\cap U})]\\&=[(V,t)]+[(U,s)].\end{aligned}$$

- *Identity element:* $[(X,0_{\mathcal{F}(X)})]$ is the zero element for $(\mathcal{F}_p,+)$. In fact,

$$\begin{aligned}&[(U,s)]+[(X,0_{\mathcal{F}(X)})]\\&=[(U\cap X, s|_{U\cap X}+0_{\mathcal{F}(X)}|_{U\cap X})]\\&=[(U,s|_U)]\\&=[(U,s)].\end{aligned}$$

- *Every element has its inverse:* The inverse element of $[(U,s)]$ is $[(U,-s)]$. For,

$$\begin{aligned}&[(U,s)]+[(U,-s)]\\&=[(U\cap U, s|_{U\cap U}+(-s)|_{U\cap U})]\\&=[(U,s-s)]=[(U,0_{\mathcal{F}(U)})]\\&=[(X,0_{\mathcal{F}(X)})].\end{aligned}$$

Henceforth, $\mathcal{F}_p$ has naturaly a structure of an abelian group.

Definition 2.4. With notation as above. The *stalk of* $\mathcal{F}$ *at* p is the abelian group $(\mathcal{F}_p,+)$.

From now on, if U is an open subset of X containing p, and s is a section of $\mathcal{F}$ over U, then we simply denote $[(U,s)]$ by s_p; and refer to s_p as the *germ* of s at p. Moreover, one may observe that there exists a natural homomorphism of abelian groups between $\mathcal{F}(U)$ and $\mathcal{F}_p$- In fact, we may consider the map given by:

$$\begin{array}{rcl}\gamma_p^U:\mathcal{F}(U) & \to & \mathcal{F}_p\\ s & \mapsto & s_p-\end{array}$$

Obviously, γ_p^U is a well-defined map. And, if s,t are sections of $\mathcal{F}$ over U, then

$$\gamma_p^U(s+t)=[(U,s+t)]=[(U,s)]+[(U,t)]=\gamma_p^U(s)+\gamma_p^U(t).$$

The following result gives one of the most important features of the concept of stalk when dealing with sheaves.

Proposition 2.5. *Let $\mathcal{F}$ be a sheaf on a topological space X, and let U be an open subset of X. If s and t are sections of $\mathcal{F}$ over U, then the following statements are equivalent:*

1. $s = t$.

2. $s_p = t_p$*, for every* $p \in U$.

Proof. If U is empty, then there is nothing to prove. And, we need only to be sure that two implies one. Let $p \in U$. Since the germ $[(U, s)]$ of s at p is equal to the germ $[(U, t)]$ of t at p (by hypothesis), there exists an open subset W_p of X containing p such that $W_p \subseteq U$ and $s|_{W_p} = t|_{W_p}$. So, $(s-t)|_{W_p} = 0_{\mathcal{F}(W_p)}$. Therefore, we found an open covering $(W_p)_{p\in U}$ of U such that the section $(s-t)$ of $\mathcal{F}$ over U has the property $(s-t)|_{W_p} = 0_{\mathcal{F}(W_p)}$ for every $p \in U$. From the fact that $\mathcal{F}$ is a sheaf, it follows that $s - t = 0_{\mathcal{F}(U)}$, that is, s and t are equals. So, we are done. □

Finally, we end this subsection with the following needed definition.

Definition 2.6. Let $\mathcal{F}$ be a presheaf on a topological space X. A *subpresheaf* of $\mathcal{F}$ is a presheaf $\mathcal{G}$ on X such that $\mathcal{G}(U)$ is a subgroup of $\mathcal{F}(U)$ for every open subset U of X, and the restrictions of $\mathcal{G}$ come from restrictions of $\mathcal{F}$.

2.1.2 Morphisms

Here, we handle a way which decides when two presheaves (respectively, sheaves) can be considered as the same. This leads to the concept of morphisms, and their induced and related data.

Definition 2.7. Let $\mathcal{F}$ and $\mathcal{G}$ be presheaves on a topological space X. A *morphism* $\varphi : \mathcal{F} \to \mathcal{G}$ of presheaves assigns a group homomorphism $\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ for every open subset U of X such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{F}(V) & \xrightarrow{\varphi_V} & \mathcal{G}(V) \\ \Big\downarrow {\scriptstyle \rho_{\mathcal{F}}{}^{V}_{W}} & & \Big\downarrow {\scriptstyle \rho_{\mathcal{G}}{}^{V}_{W}} \\ \mathcal{F}(W) & \xrightarrow[\varphi_W]{} & \mathcal{G}(W) \end{array} ,$$

for every open subsets V and W of X with $W \subseteq V$.

Given a morphism $\varphi : \mathcal{F} \to \mathcal{G}$ of presheaves. It induces a group homomorphisms between the staks. For, let p be an element of X, we may define the following map:

$$\begin{array}{cccc} \varphi_p : & \mathcal{F}_p & \mapsto & \mathcal{G}_p \\ & [(U, s)] & \mapsto & [(U, \varphi_U(s))]. \end{array}$$

Let us first prove that such map is well-defined. For, let U and V be open subsets of X containing p, and let $s \in \mathcal{F}(U)$ and $t \in \mathcal{F}(V)$ such that $[(U, s)] = [(V, t)]$. We need to be sure that $\varphi_p([(U,s)]) = \varphi_p([(V,t)])$, i.e., $[(U, \varphi_U(s))] = [(V, \varphi_V(t))]$. Indeed, by hypothesis, there exists an open subset W of X containing p with $W \subseteq U \cap V$ and $\rho_{\mathcal{F}}{}^U_W(s) = \rho_{\mathcal{F}}{}^V_W(t)$. From tha fact that φ is a morphism, for every open subset Z of X such that $W \subseteq Z$, the following diagram is commutative:

$$\begin{array}{ccc} \mathcal{F}(Z) & \xrightarrow{\varphi_Z} & \mathcal{G}(Z) \\ \Big\downarrow{\rho_{\mathcal{F}}{}^Z_W} & & \Big\downarrow{\rho_{\mathcal{G}}{}^Z_W} \\ \mathcal{F}(W) & \xrightarrow[\varphi_W]{} & \mathcal{G}(W) \end{array} .$$

Hence, $\rho_{\mathcal{G}}{}^U_W \circ \varphi_U = \varphi_W \circ \rho_{\mathcal{F}}{}^U_W$ (respectively, $\rho_{\mathcal{G}}{}^V_W \circ \varphi_V = \varphi_W \circ \rho_{\mathcal{F}}{}^V_W$), since U (respectively, V) is an open subset of X containing W. As a consequence, the following equalities hold:

$$\rho_{\mathcal{G}}{}^U_W \circ \varphi_U(s) = \varphi_W \circ \rho_{\mathcal{F}}{}^U_W(s) = \varphi_W \circ \rho_{\mathcal{F}}{}^V_W(t) = \rho_{\mathcal{G}}{}^V_W \circ \varphi_V(t).$$

Therefore, W ensures the equality $[(U, \varphi_U(s))] = [(V, \varphi_V(t))]$, and consequently, φ_p is well-defined. Next, we show that φ_p is a group homomorphism. For, without loss of generality, we may assume that the sections s and t of $\mathcal{F}$ are defined over the same open subset U of X and such that p belongs to U. So, we get the equalities below.

$$\begin{aligned} \varphi_p\big([(U,s)] + [(U,t)]\big) &= \varphi_p\big([(U, s+t)]\big) \\ &= [(U, \varphi_U(s+t))] \\ &= [(U, \varphi_U(s))] + [(U, \varphi_U(t))] \\ &= \varphi_p\big([(U,s)]\big) + \varphi_p\big([(U,t)]\big). \end{aligned}$$

Thus, φ_p is a group homomorphism.

Remark 2.8. With notation as above.

1. Let U be an open subset of X containing p. The following diagram commutes by construction:

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\varphi_U} & \mathcal{G}(U) \\ \Big\downarrow{\gamma_{\mathcal{F}}{}^U_p} & & \Big\downarrow{\gamma_{\mathcal{G}}{}^U_p} \\ \mathcal{F}_p & \xrightarrow[\varphi_p]{} & \mathcal{G}_p \end{array}$$

2. If $\mathcal{G}$ is a subpresheaf of $\mathcal{F}$, then there is naturally a well-defined morphism $j^{\#} : \mathcal{G} \to \mathcal{F}$ given by the inclusion, that is, $j^{\#}_V : \mathcal{G}(V) \to \mathcal{F}(V)$ is the inclusion for every open subset V of X. Moreover, the induced homomorphism $j^{\#}_p : \mathcal{G}_p \to \mathcal{F}_p$ is injective for every $p \in X$. Consequently, $\mathcal{G}_p$ can be considered as a subgroup of $\mathcal{F}_p$, for every $p \in X$.

Now, we deal with the concept of equality of presheaves.

Definition 2.9. Let $\varphi : \mathcal{F} \to \mathcal{G}$ and $\psi : \mathcal{G} \to \mathcal{H}$ be morphisms of presheaves on a topological space X.

1. The *composition of* φ *and* ψ is the morphism $\psi \circ \varphi : \mathcal{F} \to \mathcal{H}$ given by the data: $(\psi \circ \varphi)_U = \psi_U \circ \varphi_U$, for every open subset U of X.

2. φ is an *isomorphism* if there exists a morphism $\xi : \mathcal{G} \to \mathcal{F}$ of presheaves on X such that $\varphi \circ \xi = \mathrm{id}_{\mathcal{G}}$ and $\xi \circ \varphi = \mathrm{id}_{\mathcal{F}}$. In such case, $\mathcal{F}$ and $\mathcal{G}$ are called *isomorphic*, and are denoted by $\mathcal{F} \cong \mathcal{G}$. Here, $\mathrm{id}_{\mathcal{E}} : \mathcal{E} \to \mathcal{E}$ is given by the identity map $\mathrm{id}_{\mathcal{E}(U)} : \mathcal{E}(U) \to \mathcal{E}(U)$ for every open set U of X, and for any presheaf $\mathcal{E}$ on X.

Definition 2.10. Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves on a topological space X. φ is *injective* (respectively, *surjective*) if φ_p is injective (respectively, surjective), for every $p \in X$.

Lemma 2.11. *Let* $\varphi : \mathcal{F} \to \mathcal{G}$ *and* $\psi : \mathcal{G} \to \mathcal{H}$ *be morphisms of presheaves on a topological space* X*. Then,* $\psi_p \circ \varphi_p = (\psi \circ \varphi)_p$*, for every* $p \in X$*.*

Proof. Let U be an open subset of X containing p and s a section of $\mathcal{F}$ over U. The following equalities are straightforward:

$$\begin{aligned}
\psi_p \circ \varphi_p\big([(U,s)]\big) &= \psi_p\big([(U,\varphi_U(s))]\big)\\
&= [(U,\psi_U \circ \varphi_U(s))]\\
&= [(U,(\psi \circ \varphi)_U(s))]\\
&= (\psi \circ \varphi)_p\big([(U,s)]\big).
\end{aligned}$$

□

The next result offers a local characterization of isomorphisms of sheaves.

Theorem 2.12. *Let* X *be a topological space and* $\varphi : \mathcal{F} \to \mathcal{G}$ *a morphism of presheaves on* X*. The following statements are equivalent:*

1. φ *is an isomorphism.*

2. $\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ *is an isomorphism, for every open subset* U *of* X*.*

Furthermore, if $\mathcal{F}$ *and* $\mathcal{G}$ *are sheaves, then the above statements are equivalent to:*

3. $\varphi_p : \mathcal{F}_p \to \mathcal{G}_p$ *is an isomorphism, for every* $p \in X$*.*

Proof. The statements 1 and 2 are obviously equivalents using only the definitions. Now, let $\mathcal{F}$ and $\mathcal{G}$ be sheaves on X. Let us prove firstly that 2 implies 3. For, let p be an element of X and U an open subset of X containing p. Let s be a section of $\mathcal{F}$ over U such that $[(U,s)] \in \operatorname{Ker}\varphi_p$. From the equality $[(U,\varphi_U(s))] = [(X, 0_{\mathcal{G}(X)})]$, there exists an open subset W of X containing p with $W \subseteq U$, and such that $\rho_{\mathcal{G}}{}^U_W \circ \varphi_U(s) = 0_{\mathcal{G}(W)}$. Since φ is a morphism, the equality $\varphi_W \circ \rho_{\mathcal{F}}{}^U_W = \rho_{\mathcal{G}}{}^U_W \circ \varphi_U$ holds. Thus, $\varphi_W \circ \rho_{\mathcal{F}}{}^U_W(s) = \rho_{\mathcal{G}}{}^U_W \circ \varphi_U(s) = 0_{\mathcal{G}(W)}$. So, $\rho_{\mathcal{F}}{}^U_W(s) = 0_{\mathcal{F}(W)}$ (since φ_W is injective). Hence, $[(U,s)] = [(W, \rho_{\mathcal{F}}{}^U_W(s))] = [(W, 0_{\mathcal{F}(W)})] = [(X, 0_{\mathcal{F}(X)})]$. Next, we prove that φ_p is surjective. Indeed, let U be an open subset of X containing p, and t a section of $\mathcal{G}$ over U. Since φ_U is surjective, there exists a section s of $\mathcal{F}$ over U such that $\varphi_U(s) = t$. Then, $\varphi_p\big([(U,s)]\big) = [(U,t)]$. So, we are done.
Here, we prove that 3 implies 2. Let U be a non-empty subset of X. Let us show that φ_U is injective. For, let s be a section of $\mathcal{F}$ over U such that $s \in \operatorname{Ker}\varphi_U$ and $p \in U$. From the fact that $\varphi_U(s) = 0_{\mathcal{G}(U)}$, it follows that $\big(\varphi_U(s)\big)_p = [(U, \varphi_U(s))] = 0_{\mathcal{G}_p}$. And, since $\varphi_p(s_p) = [(U,\varphi_U(s))] = 0_{\mathcal{G}_p}$, we get $s_p = 0_{\mathcal{F}_p}$ by the injectivity of φ_p. Thus, by Proposition 2.5, $s = 0_{\mathcal{F}(U)}$. Hence, φ_U is injective as claimed.
Let us show the surjectivity of φ_U. For, let $t \in \mathcal{G}(U)$ and $p \in U$. Since φ_p is surjective, there exists an open subset U_p of X containing p and $s(p) \in \mathcal{F}(U_p)$ such that $\varphi_p\big([(U_p, s(p))]\big) = t_p$, we may assume (without lost of generality) that U_p is contained in U. Thus, from the equality $[(U_p, \varphi_{U_p}(s(p)))] = [(U,t)]$, it follows the existence of an open subset W_p of X containing p such that $W_p \subseteq U_p$ and moreover, $\rho_{\mathcal{G}}{}^{U_p}_{W_p} \circ \varphi_{U_p}(s(p)) = \rho_{\mathcal{G}}{}^U_{W_p}(t)$. Hence, since φ is a morphism, we get $\varphi_{W_p} \circ \rho_{\mathcal{F}}{}^{U_p}_{W_p}(s(p)) = \rho_{\mathcal{G}}{}^{U_p}_{W_p} \circ \varphi_{U_p}(s(p)) = \rho_{\mathcal{G}}{}^U_{W_p}(t)$. So, we obtain an open covering $(W_p)_{p\in U}$ of U with a family $\big(\rho_{\mathcal{F}}{}^{U_p}_{W_p}(s(p))\big)_{p\in U}$ of sections of $\mathcal{F}$ such that $\rho_{\mathcal{F}}{}^{U_p}_{W_p}(s(p)) \in \mathcal{F}(W_p)$ for every $p \in U$. Thus, what is left is to be sure that such family of sections can be extended to a section of $\mathcal{F}$ over U. Indeed, let p and q be elements of U. We would like to show that the equality $\rho_{\mathcal{F}}{}^{W_p}_{W_p\cap W_q}\big(\rho_{\mathcal{F}}{}^{U_p}_{W_p}(s(p))\big) = \rho_{\mathcal{F}}{}^{W_q}_{W_p\cap W_q}\big(\rho_{\mathcal{F}}{}^{U_q}_{W_q}(s(q))\big)$ holds true. To do so, it is worth noting that the following equalities occur:

$$\begin{aligned}
\varphi_{W_p\cap W_q}\big(\rho_{\mathcal{F}}{}^{W_p}_{W_p\cap W_q}\big(\rho_{\mathcal{F}}{}^{U_p}_{W_p}(s(p))\big)\big) &= \varphi_{W_p\cap W_q}\big(\rho_{\mathcal{F}}{}^{U_p}_{W_p\cap W_q}(s(p))\big)\\
&= \rho_{\mathcal{G}}{}^{U}_{W_p\cap W_q}(t)\\
&= \varphi_{W_p\cap W_q}\big(\rho_{\mathcal{F}}{}^{U_q}_{W_p\cap W_q}(s(q))\big)\\
&= \varphi_{W_p\cap W_q}\big(\rho_{\mathcal{F}}{}^{W_q}_{W_p\cap W_q}\big(\rho_{\mathcal{F}}{}^{U_q}_{W_q}(s(q))\big)\big).
\end{aligned}$$

Hence, the equality $\rho_{\mathcal{F}}{}^{W_p}_{W_p\cap W_q}\big(\rho_{\mathcal{F}}{}^{U_p}_{W_p}(s(p))\big) = \rho_{\mathcal{F}}{}^{W_q}_{W_p\cap W_q}\big(\rho_{\mathcal{F}}{}^{U_q}_{W_q}(s(q))\big)$ follows from the injectivity of $\varphi_{W_p\cap W_q}$. Now, since $\mathcal{F}$ is a sheaf, there exists $s \in \mathcal{F}(U)$ such that $\rho_{\mathcal{F}}{}^U_{W_p}(s) = \rho_{\mathcal{F}}{}^{U_p}_{W_p}(s(p))$, for every $p \in U$. Finally, let us show that $\varphi_U(s) = t$. For, since φ is a morphism, we get $\rho_{\mathcal{G}}{}^U_{W_p} \circ \varphi_U(s) = \varphi_{W_p} \circ \rho_{\mathcal{F}}{}^U_{W_p}(s) = \rho_{\mathcal{G}}{}^U_{W_p}(t)$, for all $p \in U$, i.e., $\rho_{\mathcal{G}}{}^U_{W_p}(\varphi_U(s) - t) = 0_{\mathcal{G}(W_p)}$. Therefore, from the facts that $U = \bigcup_{p\in U} W_p$

and $\mathcal{G}$ is a sheaf, it follows that $\varphi_U(s) - t = 0_{\mathcal{G}(U)}$, and consequently, we deduce that $\varphi_U(s) = t$, proving thus the surjectivity of φ_U. □

Using the same technique given in the proof of the last theorem, we obtain the following result whose proof is left to the reader:

Proposition 2.13. *Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves on a topological space X.*

1. *φ is injective, if and only if, φ_U is injective for every open subset U of X.*
2. *If φ_U is surjective for every open subset U of X, then φ is surjective.*
3. *φ is an isomorphism, if and only if, φ is injective and surjective.*

Here comes a simple way for constructing morphisms of sheaves.

Proposition 2.14. *Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves on a topological space X, and let $\mathcal{B}$ be a basis for the topology of X. If for every $B \in \mathcal{B}$, there is a homomorphism $\varphi_B : \mathcal{F}(B) \to \mathcal{G}(B)$ of abelian groups such that for every $B_1, B_2 \in \mathcal{B}$ with $B_2 \subseteq B_1$, the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{F}(B_1) & \xrightarrow{\varphi_{B_1}} & \mathcal{G}(B_1) \\ {\scriptstyle \rho_{\mathcal{F}}{}^{B_1}_{B_2}} \downarrow & & \downarrow {\scriptstyle \rho_{\mathcal{G}}{}^{B_1}_{B_2}} \\ \mathcal{F}(B_2) & \xrightarrow[\varphi_{B_2}]{} & \mathcal{G}(B_2), \end{array}$$

then, there exists a unique morphism $\psi : \mathcal{F} \to \mathcal{G}$ of sheaves on X such that $\psi_B = \varphi_B$, for all $B \in \mathcal{B}$.

Proof. Let U be a non-empty open subset of X. Since $\mathcal{B}$ is a basis for the topology of X, there exists a set I_U such that $U = \bigcup_{i \in I_U} B_i$, where $B_i \in \mathcal{B}$ for every $i \in I_U$. Let s be a section of $\mathcal{F}$ over U, and $i \in I_U$. From the fact that $\rho_{\mathcal{F}}{}^{U}_{B_i}(s)$ is a section of $\mathcal{F}$ ove B_i, it follows that $\varphi_{B_i}(\rho_{\mathcal{F}}{}^{U}_{B_i}(s))$ is a section of $\mathcal{G}$ over B_i. Thus, we construct a family $\left(\varphi_{B_i}(\rho_{\mathcal{F}}{}^{U}_{B_i}(s))\right)_{i \in I_U}$ of sections of $\mathcal{G}$ such that $\varphi_{B_i}(\rho_{\mathcal{F}}{}^{U}_{B_i}(s)) \in \mathcal{G}(B_i)$, for all $i \in I_U$. Now, we proceed to prove that such family can be extended to a section of $\mathcal{G}$ over U. Indeed, Let $i, j \in I_U$, we need to check that $\rho_{\mathcal{G}}{}^{B_i}_{B_i \cap B_j}\left(\varphi_{B_i}(\rho_{\mathcal{F}}{}^{U}_{B_i}(s))\right)$ is equal to $\rho_{\mathcal{G}}{}^{B_j}_{B_i \cap B_j}\left(\varphi_{B_j}(\rho_{\mathcal{F}}{}^{U}_{B_j}(s))\right)$. For, if $B_i \cap B_j = \emptyset$, then we are done. Hence, we may assune that $B_i \cap B_j \neq \emptyset$, so let us take $p \in B_i \cap B_j$. Therefore, there exists $B_{k_p} \in \mathcal{B}$ such that $p \in B_{k_p}$ and $B_{k_p} \subseteq B_i \cap B_j$. Using the hypothesis on the homomorphisms for the open subsets of the basis, we infer the following equalities:

$$\begin{aligned}
\rho_{\mathcal{G}}{}^{B_i\cap B_j}_{B_{k_p}} \circ \rho_{\mathcal{G}}{}^{B_i}_{B_i\cap B_j} \circ \varphi_{B_i} \circ \rho_{\mathcal{F}}{}^{U}_{B_i}(s) &= \rho_{\mathcal{G}}{}^{B_i}_{B_{k_p}} \circ \varphi_{B_i} \circ \rho_{\mathcal{F}}{}^{U}_{B_i}(s) \\
&= \varphi_{B_{k_p}} \circ \rho_{\mathcal{F}}{}^{B_i}_{B_{k_p}} \circ \rho_{\mathcal{F}}{}^{U}_{B_i}(s) \\
&= \varphi_{B_{k_p}} \circ \rho_{\mathcal{F}}{}^{U}_{B_{k_p}}(s) \\
&= \varphi_{B_{k_p}} \circ \rho_{\mathcal{F}}{}^{B_j}_{B_{k_p}} \circ \rho_{\mathcal{F}}{}^{U}_{B_j}(s) \\
&= \rho_{\mathcal{G}}{}^{B_j}_{B_{k_p}} \circ \varphi_{B_j} \circ \rho_{\mathcal{F}}{}^{U}_{B_j}(s) \\
&= \rho_{\mathcal{G}}{}^{B_i\cap B_j}_{B_{k_p}} \circ \rho_{\mathcal{G}}{}^{B_j}_{B_i\cap B_j} \circ \varphi_{B_j} \circ \rho_{\mathcal{F}}{}^{U}_{B_j}(s).
\end{aligned}$$

Thus, the last equality implies the next one:

$$\rho_{\mathcal{G}}{}^{B_i\cap B_j}_{B_{k_p}}\left(\rho_{\mathcal{G}}{}^{B_i}_{B_i\cap B_j}\left(\varphi_{B_i}(\rho_{\mathcal{F}}{}^{U}_{B_i}(s))\right) - \rho_{\mathcal{G}}{}^{B_j}_{B_i\cap B_j}\left(\varphi_{B_j}(\rho_{\mathcal{F}}{}^{U}_{B_j}(s))\right)\right) = 0_{\mathcal{G}(B_{k_p})}.$$

On the other hand, since $B_i \cap B_j = \bigcup_{p\in B_i\cap B_j} B_{k_p}$, and $\mathcal{G}$ is a sheaf, the last equation gives the claimed equality. Moreover, from the facts that $U = \bigcup_{i\in I_U} B_i$ and $\mathcal{G}$ is a sheaf, there exists a section of $\mathcal{G}$ over U, denoted by $\psi_U(s)$, such that $\rho_{\mathcal{G}}{}^{U}_{B_i}(\psi_U(s))$ is equal to $\varphi_{B_i}(\rho_{\mathcal{F}}{}^{U}_{B_i}(s))$, for every $i \in I_U$. Henceforth, we are able to define ψ_U as follows.

$$\begin{aligned}
\psi_U : \mathcal{F}(U) &\to \mathcal{G}(U) \\
s &\mapsto \psi_U(s),
\end{aligned}$$

where $\psi_U(s)$ is the section that we have just obtained, for every $s \in \mathcal{F}(U)$. This map is well-defined. Indeed, let $s, t \in \mathcal{F}(U)$ such that $s = t$, i.e., $s - t = 0_{\mathcal{F}(U)}$. It is worth noting that for every $i \in I_U$, the following equalities occur:

$$\rho_{\mathcal{G}}{}^{U}_{B_i}(\psi_U(s) - \psi_U(t)) = \varphi_{B_i}(\rho_{\mathcal{F}}{}^{U}_{B_i}(s)) - \varphi_{B_i}(\rho_{\mathcal{F}}{}^{U}_{B_i}(t)) = \varphi_{B_i}(\rho_{\mathcal{F}}{}^{U}_{B_i}(s - t)) = 0_{\mathcal{G}(B_i)}.$$

Thus, $\psi_U(s) - \psi_U(t) = 0_{\mathcal{G}(U)}$ (since $U = \bigcup_{i\in I_U} B_i$, and $\mathcal{G}$ is a sheaf), that is, $\psi_U(s)$ is equal to $\psi_U(t)$.
Next, we will prove that ψ_U is a group homomorphism. For, let s and t be sections of $\mathcal{F}$ over U. For every $i \in I_U$, we have:

$$\begin{aligned}
&\rho_{\mathcal{G}}{}^{U}_{B_i}(\psi_U(s+t) - \psi_U(s) - \psi_U(t)) \\
&= \varphi_{B_i}(\rho_{\mathcal{F}}{}^{U}_{B_i}(s+t)) - \varphi_{B_i}(\rho_{\mathcal{F}}{}^{U}_{B_i}(s)) - \varphi_{B_i}(\rho_{\mathcal{F}}{}^{U}_{B_i}(t)) \\
&= \varphi_{B_i}(\rho_{\mathcal{F}}{}^{U}_{B_i}(s + t - s - t)) \\
&= 0_{\mathcal{G}(B_i)}.
\end{aligned}$$

So, $\psi_U(s + t) - \psi_U(s) - \psi_U(t) = 0_{\mathcal{G}(U)}$ (since $U = \bigcup_{i\in I_U} B_i$ and $\mathcal{G}$ is a sheaf). Consequently, $\psi_U(s + t) = \psi_U(s) + \psi_U(t)$. Hence, ψ_U is a group homomorphism. Now, let us prove that the family $(\psi_U)_{\{U \text{ is an open subset of} X\}}$ is a morphism that we will denote by ψ. For, let U and V be open subsets of X such that $V \subseteq U$. Here, we show the commutativity of the following diagram:

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\psi_U} & \mathcal{G}(U) \\ \downarrow {\scriptstyle \rho_{\mathcal{F}}{}_V^U} & & \downarrow {\scriptstyle \rho_{\mathcal{G}}{}_V^U} \\ \mathcal{F}(V) & \xrightarrow[\psi_V]{} & \mathcal{G}(V), \end{array}$$

that is, we need to check that the equality $\rho_{\mathcal{G}}{}_V^U \circ \psi_U = \psi_V \circ \rho_{\mathcal{F}}{}_V^U$ is true. Indeed, let s be a section of $\mathcal{F}$ over U. It is worth noting that for every $i \in I_V$, the following equalities are satisfied:

$$\begin{aligned} \rho_{\mathcal{G}}{}_{B_i}^V\big(\rho_{\mathcal{G}}{}_V^U \circ \psi_U(s) - \psi_V \circ \rho_{\mathcal{F}}{}_V^U(s)\big) &= \rho_{\mathcal{G}}{}_{B_i}^V \circ \rho_{\mathcal{G}}{}_V^U \circ \psi_U(s) - \rho_{\mathcal{G}}{}_{B_i}^V \circ \psi_V \circ \rho_{\mathcal{F}}{}_V^U(s) \\ &= \varphi_{B_i}\big(\rho_{\mathcal{F}}{}_{B_i}^U(s)\big) - \varphi_{B_i}\big(\rho_{\mathcal{F}}{}_{B_i}^V \circ \rho_{\mathcal{F}}{}_V^U(s)\big). \\ &= 0_{\mathcal{G}(B_i)}. \end{aligned}$$

Using the fact that $V = \bigcup_{i \in I_V} B_i$ and $\mathcal{G}$ is a sheaf, we get the equality between $\rho_{\mathcal{G}}{}_V^U \circ \psi_U(s) - \psi_V \circ \rho_{\mathcal{F}}{}_V^U(s)$ and $0_{\mathcal{G}(U)}$, i.e., $\rho_{\mathcal{G}}{}_V^U \circ \psi_U(s)$ is equal to $\psi_V \circ \rho_{\mathcal{F}}{}_V^U(s)$. Thus, ψ is a morphism. Finally, we have to be sure that if $B \in \mathcal{B}$, then $\psi_B = \varphi_B$. In fact, let $B \in \mathcal{B}$, and let s be a section of $\mathcal{F}$ over B. By construction of ψ_B, the following equalities hold:

$$\psi_B(s) = \rho_{\mathcal{G}}{}_B^B(\psi_B(s)) = \varphi_B(\rho_{\mathcal{F}}{}_B^B(s)) = \varphi_B(s).$$

Therefore, $\psi_B = \varphi_B$. So, we are done. □

Corollary 2.15. *With notation and hypothesis as in the last proposition. If φ_B is an isomorphism, for every $B \in \mathcal{B}$, then $\mathcal{F}$ and $\mathcal{G}$ are isomorphic.*

Proof. By Proposition 2.14, there exists naturally a morphism $\psi : \mathcal{F} \to \mathcal{G}$ of sheaves on X. So, we need only to prove that ψ is an isomorphism. To do so, we prove that the induced homomorphism between the stalks $\psi_p : \mathcal{F}_p \to \mathcal{G}_p$ is an isomorphism, for every $p \in X$. For, let $p \in X$.

Firstly, we deal with the injectivity of ψ_p. So, let U be an open subset of X containing p, and s be a section of $\mathcal{F}$ over U such that $[(U, s)] \in \operatorname{Ker} \psi_p$. Since $\mathcal{B}$ is a basis for the topology of X, there exists $B_1 \in \mathcal{B}$ such that $p \in B_1$, and $B_1 \subseteq U$. From the equality $[(B_1, \psi_{B_1}(\rho_{\mathcal{F}}{}_{B_1}^U(s)))] = [(X, 0_{\mathcal{G}(X)})]$, we deduce the existence of an open subset W of X containing p, moreover is contained in B_1, and $\rho_{\mathcal{G}}{}_W^{B_1}\big(\psi_{B_1}(\rho_{\mathcal{F}}{}_{B_1}^U(s))\big) = 0_{\mathcal{G}(W)}$. Using again the fact that $\mathcal{B}$ is a basis for the topology of X, we infer the existence of $B_2 \in \mathcal{B}$ such that $p \in B_2$, and $B_2 \subseteq W$. Now, the equality $\rho_{\mathcal{G}}{}_W^{B_1}\big(\psi_{B_1}(\rho_{\mathcal{F}}{}_{B_1}^U(s))\big) = 0_{\mathcal{G}(W)}$ implies $\rho_{\mathcal{G}}{}_{B_2}^{B_1}\big(\psi_{B_1}(\rho_{\mathcal{F}}{}_{B_1}^U(s))\big) = 0_{\mathcal{G}(B_2)}$. And, since ψ is a morphism, we get

$\psi_{B_2}\big(\rho_{\mathcal{F}}{}^{B_1}_{B_2} \circ \rho_{\mathcal{F}}{}^{U}_{B_1}(s)\big) = 0_{\mathcal{G}(B_2)}$. Hence, $\rho_{\mathcal{F}}{}^{U}_{B_2}(s) = 0_{\mathcal{F}(B_2)}$ (since ψ_{B_2} is injective), and consequently we obtain:

$$[(U, s)] = [(B_2, \rho_{\mathcal{F}}{}^{U}_{B_2}(s))] = [(B_2, 0_{\mathcal{F}(B_2)})] = 0_{\mathcal{F}_p}.$$

Therefore, ψ_p is injective.
Secondly, we take care of the surjectivity of ψ_p. Indeed, let U be an open subset of X containing p, and let t be a section of $\mathcal{G}$ over U. We may take $[(U, t)]$ of $\mathcal{G}_p$. Since $\mathcal{B}$ is a basis for the topology of X, there exists $B \in \mathcal{B}$ with $p \in B$, and $B \subseteq U$. Thus, from the surjectivity of ψ_B, we get the existence of a section s of $\mathcal{F}$ over B such that $\psi_B(s) = \rho_{\mathcal{G}}{}^{U}_{B}(t)$. And, consequently, the following equalities:

$$\psi_p\big([(B, s)]\big) = [(B, \psi_B(s))] = [(B, \rho_{\mathcal{G}}{}^{U}_{B}(t))] = [(U, t)],$$

prove the surjectivity of ψ_p.
Therefore, ψ_p is an isomorphism for every $p \in X$. Hence, ψ is an isomorphism of sheaves. □

2.2 Very Usefull Sheaves

In this subsection, we deal with some important sheaves, and study their properties.

2.2.1 Restriction of a Sheaf to any Open Subset

Let $\mathcal{F}$ be a presheaf on a topological space X, and let U be a non-empty open set of X. U inherits a structure of a topological space that comes from the topology of X, known as, the *induced topology*. Next, our task is to define naturally a presheaf $\mathcal{F}|_U$ on U. For, let V be an open subset of U, and let W be an open subset of U such that $W \subseteq V$, we define $\mathcal{F}|_U(V) = \mathcal{F}(V)$, and $\rho_{\mathcal{F}|_U}{}^{V}_{W} = \rho_{\mathcal{F}}{}^{V}_{W}$. From the facts that U is an open subset of X, and $\mathcal{F}$ is a presheaf on X, it follows obviously that $\mathcal{F}|_U$ is a presheaf on U. Furthermore, if $\mathcal{F}$ is a sheaf, then $\mathcal{F}|_U$ is also a sheaf. This justifies the following:

Definition 2.16. With notation as above, $\mathcal{F}|_U$ is the *restriction of the sheaf* $\mathcal{F}$ *to* U.

Now, we are interested in the stalks of this sheaf. So, let p be a point of U. We may consider the following map:

$$\begin{array}{rcl} \varphi_p : (\mathcal{F}|_U)_p & \to & \mathcal{F}_p \\ {[(V, s)]} & \mapsto & [(V, s)]. \end{array}$$

It is easy to see that φ_p is a group homomorphism. Moreover, it is an isomorphism. Indeed,

- φ_p is injective. For, let V be an open subset of U containing p, and let s be a section of $\mathcal{F}|_U$ over V such that $[(V,s)] \in \operatorname{Ker}\varphi_p$. The equality $[(V,s)] = [(X, 0_{\mathcal{F}(X)})]$ implies the existence of an open subset Z of X containing p such that $Z \subseteq V$, and $\rho_{\mathcal{F}}{}^V_Z(s) = 0_{\mathcal{F}(Z)}$. We may observe that $Z \cap U$ is an open subset of U containing p and is contained in V, and consequently, the following equalities hold: $[(V,s)] = [(Z\cap U, \rho_{\mathcal{F}|_U}{}^V_{Z\cap U}(s))] = [(Z\cap U, 0_{\mathcal{F}|_U(Z\cap U)})] = [(U, 0_{\mathcal{F}|_U(U)})]$. Thus, φ_p is injective.

- φ_p is surjective. Indeed, let V be an open subset X containing p, and let s be a section of $\mathcal{F}$ over V. Since $V \cap U$ is an open subset of U containing p, we may consider the element $[(V \cap U, \rho_{\mathcal{F}|_U}{}^V_{V\cap U}(s))]$ of $(\mathcal{F}|_U)_p$. As a consequence, $\varphi_p\big([(V \cap U, \rho_{\mathcal{F}|_U}{}^V_{V\cap U}(s))]$ and $[(V,s)]$ are equals. Therefore, φ_p is surjective.

This study leads to:

Proposition 2.17. *With notation as above. $(\mathcal{F}|_U)_p$ is isomorphic to $\mathcal{F}_p$, for every $p \in U$.*

Using the restriction of sheaves, the following result offers a way to detect the surjectivity of a given morphism of sheaves:

Lemma 2.18. *Let $\varphi : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves on a topological space X. φ is surjective, if and only if, the morphism $\varphi|_U : \mathcal{F}|_U \to \mathcal{G}|_U$ is surjective, for every open subset U of X.*

Proof. Let us assume that $\varphi|_U : \mathcal{F}|_U \to \mathcal{G}|_U$ is surjective, for every open subset U of X. The particular case where the open subset X is X itself, we get that the morphism $\varphi|_X : \mathcal{F}|_X \to \mathcal{G}|_X$ is surjective, and from the fact that $\varphi|_X = \varphi$, $\mathcal{F}|_X = \mathcal{F}$, and $\mathcal{G}|_X = \mathcal{G}$, we deduce that $\varphi : \mathcal{F} \to \mathcal{G}$ is surjective.
Conversely, let us assume that the morphism $\varphi : \mathcal{F} \to \mathcal{G}$ is surjective. Let U be an open subset of X. If $U = \emptyset$, then there nothing to prove, since all are zeros. So, we may consider U to be non-empty, and $p \in X$. It is worth nothing that the following diagram is commutative:

$$
\begin{array}{ccc}
(\mathcal{F}|_U)_p & \xrightarrow{(\varphi|_U)_p} & (\mathcal{G}|_U)_p \\
\downarrow \cong & & \downarrow \cong \\
\mathcal{F}_p & \xrightarrow[\varphi_p]{} & \mathcal{G}_p
\end{array}
$$

Hence, $(\varphi|_U)_p$ is the composition of three homomorphisms that are all surjectives. Thus, $(\varphi|_U)_p$ is surjective for every p in U. Consequently, $\varphi|_U$ is surjective. $\square$