BmPa

Bollettino di Matematica pura e applicata

V

The *Bollettino di Matematica Pura ed Applicata* (BmPa) is the scientific journal of the Section of Mathematics of the Dipartimento di Ingegneria Elettrica, Elettronica e delle Telecomunicazioni, di Tecnologie Chimiche, Automatica e Modelli Matematici (DIEETCAM) of the Università di Palermo. It publishes original research papers and survey papers in pure and applied mathematics, and it is open to contributions of Italian and stranger researchers. The papers emphasize the advances of knowledge in mathematics problems and new applications. All the papers are peer-reviewed. The BmPa was created in 2008 mainly to give, once in a year, an overview of the activities of the peoples working in the Dipartimento di Metodi e Modelli Matematici (DMMM). The DMMM, which has been merged to the DIEETCAM of the Università di Palermo, was created thanks to the will of the all the professors of the mathematical area of the Facoltà di Ingegneria. Both the scientific and didactic activity of the DMMM can be found in the web-site www.dmmm.unipa.it. From its constitution it has been involved in many collaborations with Italian and foreign researchers.

Editors Maria Stella Mongiovì, Michele Sciacca.

Bollettino di Matematica pura e applicata

Volume V

Editors
Maria Stella Mongiovì
Michele Sciacca

www.aracneeditrice.it
info@aracneeditrice.it

via Raffaele Garofalo, 133/A-B
00173 Roma
(06) 93781065

ISBN 978-88-548-6024-7

I edizione: maggio 2013

Examples of additive designs

Andrea Caggegi

Dipartimento di Ingegneria Elettrica, Elettronica e delle Telecomunicazioni,
di Tecnologie Chimiche, Automatica e Modelli Matematici (DIEETCAM)
Università degli Studi di Palermo, Viale delle Scienze, 90128 Palermo, Italy
E-mail: andrea.caggegi@unipa.it

Abstract

In this paper we present some additive designs.

AMS MSC: 05B05, 05B25
Key words: Block design, additive design

By a $2-(v,k,\lambda)$ design we understand a pair $\mathcal{D}=(\mathcal{P},\mathcal{B})$ where $\mathcal{P}$ is a set of v elements (called *points*) and $\mathcal{B}$ is a collection of distinguished subsets of $\mathcal{P}$ (called *blocks*) such that each block contains precisely k points and any two distinct points are contained in exactly λ common blocks. We say that a $2-(v,k,\lambda)$ design $\mathcal{D}=(\mathcal{P},\mathcal{B})$ is an additive design if there are a finite abelian group $(G,+)$ and an injective mapping $f:\mathcal{P}\longrightarrow G$ such that $\sum_{x\in C} f(x)=0$ whenever $C\in\mathcal{B}$ is a block of $\mathcal{D}$. Having said this, we present now some additive designs.

Suppose p is an odd prime number and let q,n,m be positive integers such that: $q=p^\alpha$ is a power of p; m is divisible by p; $3\leq m\leq n+1$.

Let $\mathcal{D}=(\mathcal{P},\mathcal{B})$ be the simple incidence structure whose point-set $\mathcal{P}$ is a given n–dimensional left vector space V over F_q (the finite field with q elements) and whose block-set $\mathcal{B}$ consists of all subsets $B(x_1,x_2,\dots,x_m)$ of $\mathcal{P}$ of the form

$$B(x_1,x_2,\dots,x_m)=\{x_j+t(x_1+x_2+\dots x_m)|\,1\leq j\leq m \text{ and } t\in F_p\}$$

where $x_1,x_2,\dots,x_m\in V$ are affinely independent vectors (when one regards V as affine space over itself) and F_p is the subfield of F_q with p elements.

Suppose $Y=B(y_1,y_2,\dots,y_m)$ is a block of $\mathcal{D}=(\mathcal{P},\mathcal{B})$. Since affinely independence of vectors $y_1,y_2,\dots y_m\in V$ implies $a_1=a_2=\dots=a_m=0\in F_p$ whenever $\begin{cases} a_1y_1+a_2y_2+\dots a_my_m=0\\ a_1+a_2+\dots+a_m=0\end{cases}$, we deduce $y_1+y_2+\dots y_m\neq 0\in V$ and the block $Y=B(y_1,y_2,\dots,y_m)$ contains exactly mp points (vectors). Because $G=\mathrm{Aff}(V)$

(the affine group of V) acts 2-homogeneously on V and permutes the subsets $W = \{w_1, w_2, \ldots, w_m\}$ of V consisting of $m = |W|$ affinely independent vectors, the block-set $\mathcal{B}$ may be written as $\mathcal{B} = X^G$ (the G-orbit of the block $X = B(x_1, x_2, \ldots x_m)$) and it follows from [1, Proposition 4.6] that $\mathcal{D} = (\mathcal{P}, \mathcal{B}) = (V, X^G, \in)$ is a $2-(v,k,\lambda)$ design with parameters

$$v = q^n, k = mp, b = \frac{|G|}{|S_X|}, \lambda = \frac{k(k-1)}{v(v-1)}b, r = \frac{kb}{v}$$

where $S_X = \{f \in \mathrm{A}ff(V) |\, X^f = X\}$ is the setwise stabilizer of the base block $X = B(x_1, x_2, \ldots x_m)$ and r is the replication number of $\mathcal{D}$ (*i.e.*, r is the number of blocks through any given fixed point)

Note that such a design is an additive design since $\sum_{y \in Y} y = p(y_1 + y_2 + \cdots + y_m) = 0 \in V$ for every block $Y = B(y_1, y_2, \ldots, y_m)$ of $\mathcal{D} = (\mathcal{P}, \mathcal{B}) = (V, X^G, \in)$.

We now determine the number b of blocks of the additive 2–design $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ above described. To do this we first need to prove :

Lemma 1. *Let $Y = B(y_1, y_2, \ldots, y_m)$ be any block of the 2–design $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ and denote by R_Y the number of (unordered) sets $\{z_1, z_2, \ldots, z_m\} \subset Y$ consisting of affinely independent vectors $z_1, z_2, \ldots, z_m$ having the property that $B(z_1, z_2, \ldots, z_m) = Y$. Then R_Y does not depend on Y and we have*

$$R_Y = \begin{cases} p^{m-1}(p-1) \text{ if } m > 3 \\ 72 \text{ if } m = 3 \end{cases}$$

Proof. If $t_1, t_2, \ldots t_m \in F_p$ are chosen in such a way that $t_1 + t_2 + \ldots t_m \neq -1 \in F_p$, then

$z_1 = y_1 + t_1(y_1 + y_2 + \cdots + y_m), z_2 = y_2 + t_2(y_1 + y_2 + \cdots + y_m), \ldots,$
$z_m = y_m + t_m(y_1 + y_2 + \cdots + y_m)$

are m distinct vectors of V, affinely independent (belonging to Y and) having the property that $B(z_1, z_2, \ldots z_m) = Y$: hence $R_Y \geq p^m - p^{m-1} = p^{m-1}(p-1)$. On the other hand since

$$l_j = \{y_j + \tau(y_1 + y_2 + \cdots + y_m) |\, \tau \in F_q\} \qquad (j = 1, 2, \ldots m)$$

are m distinct parallel lines of V such that $Y \subseteq l_1 \cup l_2 \cup \cdots \cup l_m$, we infer: if $m > 3$ and if $B(w_1, w_2, \ldots, w_m) = Y$ for suitable affinely independent vectors $w_1, w_2, \ldots w_m \in Y$, then the m-set $\{w_1, w_2, \ldots w_m\}$ meets each line l_j $(j = 1, 2, , m)$ in just one point (vector) and so there are $c_1, c_2, \ldots c_m \in F_p$ such that $w_j = y_j + c_j(y_1 + y_2 + \cdots + y_m)$ for $j = 1, 2, , m$. Therefore we must have $R_Y \leq p^{m-1}(p-1)$ if $m > 3$. Now we know that $m > 3$ implies $R_Y \leq p^{m-1}(p-1) \leq R_Y$ and this gives $R_Y = p^{m-1}(p-1)$ if $m > 3$.

Finally, suppose $m = 3$. Then $p = 3$ and Y is the finite affine plane of order 3 whose points are

$$y_1, y_2, y_3, -y_1 + y_2 + y_3, y_1 - y_2 + y_3, y_1 + y_2 - y_3, -y_2 - y_3, -y_1 - y_3, -y_1 - y_2$$

and whose lines are the following twelve 3-sets

$$y_1 + \langle y_1 + y_2 + y_3 \rangle = \{y_1 + t(y_1 + y_2 + y_3) |\, t \in F_p\} = \{y_1, -y_1 + y_2 + y_3, -y_2 - y_3\}$$
$$y_2 + \langle y_1 + y_2 + y_3 \rangle = \{y_2 + t(y_1 + y_2 + y_3) |\, t \in F_p\} = \{y_2, y_1 - y_2 + y_3, -y_1 - y_3\}$$
$$y_3 + \langle y_1 + y_2 + y_3 \rangle = \{y_3 + t(y_1 + y_2 + y_3) |\, t \in F_p\} = \{y_3, y_1 + y_2 - y_3, -y_1 - y_2\}$$

$$y_1 + \langle y_2 - y_1 \rangle = \{y_1 + t(y_2 - y_1) |\, t \in F_p\} = \{y_1, y_2, -y_1 - y_2\}$$
$$y_1+y_2-y_3+\langle y_2-y_1 \rangle = \{y_1+y_2-y_3+t(y_2-y_1) |\, t \in F_p\} = \{y_1+y_2-y_3, -y_2-y_3, -y_1-y_3\}$$
$$y_3 + \langle y_2 - y_1 \rangle = \{y_3 + t(y_2 - y_1) |\, t \in F_p\} = \{y_3, -y_1 + y_2 + y_3, y_1 - y_2 + y_3\}$$

$$y_1 + \langle y_3 - y_1 \rangle = \{y_1 + t(y_3 - y_1) |\, t \in F_p\} = \{y_1, y_3, -y_1 - y_3\}$$
$$y_2 + \langle y_3 - y_1 \rangle = \{y_2 + t(y_3 - y_1) |\, t \in F_p\} = \{y_2, -y_1 + y_2 + y_3, y_1 + y_2 - y_3\}$$
$$y_1-y_2+y_3+\langle y_3-y_1 \rangle = \{y_1-y_2+y_3+t(y_3-y_1) |\, t \in F_p\} = \{y_1-y_2+y_3, -y_2-y_3, -y_1-y_2\}$$

$$y_1 + \langle y_3 - y_2 \rangle = \{y_1 + t(y_3 - y_2) |\, t \in F_p\} = \{y_1, y_1 - y_2 + y_3, y_1 + y_2 - y_3\}$$
$$y_2 + \langle y_3 - y_2 \rangle = \{y_2 + t(y_3 - y_2) |\, t \in F_p\} = \{y_2, y_3, -y_2 - y_3\}$$
$$y_3+y_2-y_1+\langle y_3-y_2 \rangle = \{y_3+y_2-y_1+t(y_3-y_2) |\, t \in F_p\} = \{y_3+y_2-y_1, -y_1-y_3, -y_1-y_2\}.$$

Then three vectors (points) $z_1, z_2, z_3 \in Y$ are affinely independent (and have the property that $B(z_1, z_2, z_3) = Y$) if and only if z_1, z_2, z_3 are non-collinear points of (the affine plane) Y: therefore $R_Y = \binom{9}{3} - 12 = 84 - 12 = 72$ if $m = 3$. The lemma is proved. □

Since $\frac{q^n(q^n-1)(q^n-q)\cdots(q^n-q^{m-2})}{1\cdot 2\cdot 3\cdots m}$ is the number of all the m-subsets of V consisting of affinely independent vectors, counting in two ways the number of flags (W, Y), where $W = \{w_1, w_2, \ldots, w_m\}$ is an m-subset of V consisting of affinely independent vectors and $Y = B(y_1, y_2, \ldots, y_m)$ is a block of $\mathcal{D} = (\mathcal{P}, \mathcal{B}) = (V, X^G, \in)$ through W, we obtain (by Lemma 1)

$$\frac{q^n(q^n-1)(q^n-q)\cdots(q^n-q^{m-2})}{1\cdot 2\cdot 3\cdots m} = p^{m-1}(p-1)b \ \text{ if } m > 3$$

and

$$\frac{q^n(q^n-1)(q^n-q)}{6} = 72b \text{ if } m = 3$$

Therefore we find

$$b = \frac{q^n(q^n-1)(q^n-q)\cdots(q^n-q^{m-2})}{(1\cdot 2\cdot 3\cdots m)p^{m-1}(p-1)} \ \text{ if } m > 3$$

and

$$b = \frac{q^n(q^n-1)(q^n-q)}{6 \cdot 72} = \frac{q^n(q^n-1)(q^n-q)}{432} \text{ if } m = 3$$

Remark 1. As the affine group $\mathrm{Aff}(V)$ has order $|\mathrm{Aff}(V)| = q^n(q^n-1)(q^n-q)\cdots(q^n-q^{n-1})$ and $b = \frac{|\mathrm{Aff}(V)|}{|S_X|}$, we may conclude that S_X is a group of order

$$|S_X| = (1 \cdot 2 \cdot 3 \cdots m)p^{m-1}(p-1)(q^n-q^{n-1})(q^n-q^{n-2})\cdots(q^n-q^{m-1}) \text{ if } m > 3$$

and

$$|S_X| = 432(q^n-q^{n-1})(q^n-q^{n-2})\cdots(q^n-q^2) \text{ if } m = 3$$

Remark 2. Suppose $m = 3$ and $q = p$. Then $q = p = 3$ and $\mathcal{D} = (\mathcal{P}, \mathcal{B})$ is the additive 2–design whose points are the vectors of V and whose blocks are the 2–dimensinal affine subspace of V: *i.e.*, $\mathcal{P} = V$ and $\mathcal{B}$ is the set of all cosets $z + U = \{z + u |\, u \in U\}$ where $z \in V$ and U is a 2–dimensional subspace of V.

References

[1] T. Beth, D. Jungnickel, H. Lenz, Design theory 2nd ed., Cambridge University Press (1999)

[2] A. Caggegi, Some additive $2-(v, 4, \lambda)$ design, Boll. di mat. pura ed appl. Vol. II (2009), 1-3

[3] C. J. Colbourn, J. H. Dinitz , The CRC Handbook of Combinatorial Designs, CRC Press, Boca Raton (1996)