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Meshfree approximation for multi–asset European and American option problems

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Introduction

This work is intended as a study of the applicability of the meshless approach of radial basis functions approximation to the solution of the well-known *Black-Scholes* differential equation applied to single and multi assets European and American option problems.

- Firstly, we price an European call option and an European basket call option by solving the Black-Scholes equation with two different radial basis functions (RBF) approaches, and with a finite centered differences method. In order to verify the efficiency of the RBF methods, we computed the root mean square error of our approximate solutions (with respect to a reference solution) at an increasing number of grid points on which we evaluate the approximants. We test also a variant of the Black-Scholes equation, obtained by applying a change of variables, aimed to see if this produces a smaller error.

  In the case of an European call option the reference solution is given by known solution of the Black-Scholes equation. In order to decrease the root mean square error, in the numerical experiments we use two types of spatial discretizations, equispaced and not equispaced.

  In the case of European basket call options, since the explicit solution is not available, as reference solution we consider the one obtained by a central difference approximation since this method was the one that approximates the exact solution in a better way.

- Secondly, we study the solution of multi-asset American problems. We do this because these methods allow us to increase easily the number of assets without increasing drastically the computational cost. After deriving the corresponding Black-Scholes equation we describe the Radial Basis Functions (RBF) and their principal characteristics.

  The approximate solution is obtained by a penalty method which allows us to remove the free and moving boundary conditions by adding a small continuous penalty term to the Black-Scholes equation. For this purpose, we apply explicit, implicit and semi-implicit schemes for one and two factor problems and we discuss their stability.

  For American options, we do not know the exact solution of the Black-Scholes equation. Therefore, we consider as *reference solution*, the one obtained by a linearly implicit finite difference method in space. Then, as approximant, we compute the RBF solution. To integrate in the time we use the θ-method. The resulting approximation, compared with the reference solution, turns out to be quite promising since as the number of points grows the root mean square error decreases. We make this comparison for the cases of one and two underlying assets.

In both instances, the main conclusion is that the RBF approach is a promising method for approximating the solution, or the reference one, of the Black-Scholes differential equation.

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Chapter 1

The Black-Scholes equation: history

1.1 Introduction and a short history

If we look back at the world in the mid-sixties, price risk in financial markets was fairly limited. This was an era when state interventions were much more prevalent, the world over, and many kinds of risk were apparently absent, owing to price controls. Ever since the early seventies, governments all over the world have steadily retreated from overt price controls. This has been motivated by an increasing realization that these stabilization programs involve enormous costs, upon governments in particular and upon the economy in general through distortions in resource allocation. The deregulation of the four major financial markets (i.e equity, debt, foreign exchange and commodities) had many consequences for productivity and economic growth. It has also generated an upsurge of price volatility. The term "risk" is often interpreted as the probability of encountering losses. In the language of modern economics, however, risk is defined as volatility, where unexpected changes (whether in the positive or negative direction) are viewed symmetrically. Volatility in major financial markets of the world rose sharply in the early seventies. From the firm perspective, the elimination of financial risk using financial instruments is desirable, insofar as it does not alter the actual functioning of the firm. This reasoning illustrates the role for options and futures in the modern economy: these markets provide means through which economic agents can control the risks that they are exposed to. It should be emphasised that through these methods, risk is not destroyed, it is only transferred from one economic agent to another. The buyer of an option reduces his risk, but that risk is transferred to the seller of the option. The financial markets which enable this repackaging and transfer of risk are called financial derivatives markets. Financial derivatives are the modern functional replacement for the 'price stabilisation programs' which governments once used. However, financial derivatives do not eliminate risk or price volatility in the economy; instead, they give individual economic agents the means through which their risk can be transferred to others. When thousands of firms and individuals in the entire economy use derivatives, we obtain a very different distribution of risk in the economy as compared with what is found without derivatives. Since economic agents voluntarily enter into these contracts, they have to be better off as a consequence of the contracting. Thus the use of derivatives markets can only be welfare-enhancing.

The focus of the works of Black and Sholes [1] and Merton [2] are on options, because options are analytically complex. A type of option is a call. An example is the following. A person A buys a call from a person B with strike price K and maturity T. Then A has the right to buy the call's underlying at the time T, but not the obligation. A will exercise the call if and only if at the time T the stock price S is bigger then K. The *payoff diagram* of A is a graph similar to the one in Figure 1.1.

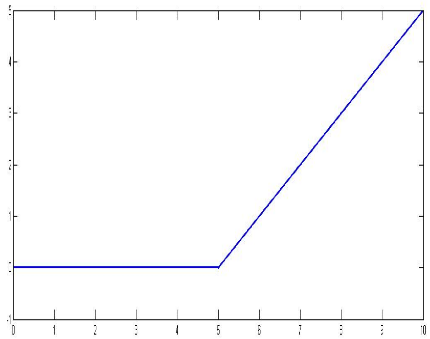

Figure 1.1: Payoff of a buying call with $K = 5$ at time T.

From the point of view of the buyer A of the option, once the price of the option is paid, an option has only upside potential - it can yield profits, but there can be no loss. Conversely, the seller of the option obtains the price of the option immediately, and then runs the risk of losses upon option exercise. The payoff diagram of the seller B of the option can be like in Figure 1.2.

The central puzzle about options concerns pricing. What is an option worth? Given the characteristics that define an option, how is its fair value determined?

The characteristics may be enumerated as:

1. the identity of the underlying asset;

2. the exercise price X;

3. the date of exercise T;

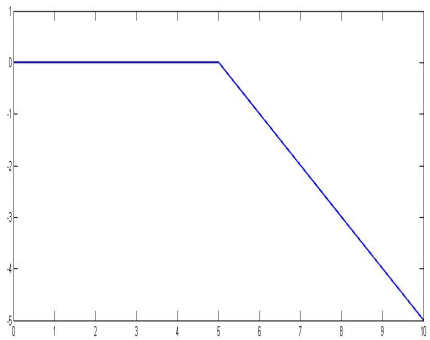

Figure 1.2: Payoff of a selling call with $K = 5$ at time T.

4. the present spot price of the underlying asset S.

Options trading has been done for many centuries by traders using their instincts to guide the choice of prices. At the dawn of modern financial economics, researchers faced the challenge of finding a scientific theory which would yield an explicit solution to the question of how options can be priced. The identity of the underlying asset impinges upon option pricing via the volatility of returns on the asset. Options on more volatile assets are more valuable, e.g. the insurance premium would be higher if there was more uncertainty about an outcome. When the volatility of an asset goes up, options on that asset become more valuable.

The first modern attempt of analysing options dates back to the year 1900, when the young French mathematician Louis Bachelier wrote a dissertation at Sorbonne titled *The Theory of Speculation*. Bachelier was the first person to think about financial prices using the modern tools of probability theory. The approach that he took, and many of the results that he obtained, were far ahead of their time. As a consequence, they were to lie dormant for sixty years. In the sixties, MIT was a hotbed of interest and curiosity about options. A host of researchers, including Paul Samuelson, worked on the question of pricing options. From a modern perspective, we can classify many of the early attempts at pricing options as being risk based models: these were option pricing formulas which required the knowledge of expected future returns on the underlying asset. The practical usefulness of these theories was limited, because forecasting the expected future returns on the underlying asset is difficult. A major advance in option pricing was accomplished by Hans Stoll in 1969, when he used the no arbitrage argument to link up the price of a call option and a put option. This principle is called "put-call parity".

This was the situation when Fischer Black, Robert C. Merton and Myron Scholes entered the picture. Black was a Ph.D. in applied mathematics, who was then working at the consulting firm Arthur D. Little Inc. Merton had studied mathematics at CalTech and then become a Ph.D. student of Paul Samuelson. Scholes was a fresh Ph.D. in economics from Chicago who had joined the faculty of the MIT finance department. The research strategy that they uncovered was, in fact, an arbitrage based argument, but it was not the simple arbitrage of the variety discussed above. In fact, a simple arbitrage argument such as that used in our gold example, or that used in obtaining put call parity, cannot be found in option pricing. These "simple" arguments rely on a single transaction, which buys what is cheap and sells what is costly, and accomplishes riskless profits no such arbitrage can be found with options. The first major insight was the idea that a dynamic arbitrage can be setup between the underlying and the call option.

In more general problems, the Black-Scholes analysis, and the field of continuous time finance, is a powerful technology for dealing with a wide variety of financial instruments. In all cases, a differential equations defining the price of the asset of interest can be derived, but usually it is not possible to find analytical solutions to these equations.

The impact upon the options market was very strong. Options have been traded informally, in what are known as "over the counter" markets, for centuries. It was as late as 1973 when the first trading of options at an exchange, which was the Chicago Board Options Exchange (CBOE). Thus there was a happy coincidence between the arrival of the analytical understanding of options and the development of market institutions to trade in options. Within just a few months after the Black Scholes paper was published, Texas Instruments started selling hand calculators which had the capability of evaluating the Black Scholes formula! Today, every MBA student in the world is taught the Black Scholes formula. As mentioned earlier, options have existed for centuries, but a major constraint upon their usefulness was the enormous difficulties of their pricing. In a world where very little was known about how options should be priced, trading options was a mixture of guesswork and gambling, and very few economic agents participated in options markets. With the analytical capabilities created by Black, Merton and Scholes, the option has become a mainstream instrument, with millions of users all over the world being able to meaningfully think about option pricing.

1.1.1 Some basic elements

The Markov process

A Markov process is a particular type of stochastic process where only the present value of a variable is relevant for predicting the future. The stock price follows a stochastic process then, we can suppose that they follows a Markov process.

If the stock price follows a Markov process, our predictions for the futures should be unaffected by the temporal way followed in the past from the stock price. The competition in the marketplace tends to ensure that weak-form market efficiency holds. The current stock price inpounds all the information contained in a record of past prices.

Wiener processes

The Wiener processes are a particular type of Markov processes with a mean of zero and a variance rate of one per year.

Definition 1. *A variable $z(t)$ follows a Wiener process if it has the properties:*

1. $z(T) - z(0) \sim N(0, T)$, *then* $dz = \epsilon\sqrt{dt}$ *whit* $\epsilon \sim N(0, 1)$
2. *moreover, if the temporal intervals are separated also the increases are independent.*

If we reduce the temporal interval to an infinitesimal unity dz we obtain $dz \sim N(0, dt)$. The properties of dz are:

- $(dz)^2 = dt$;
- $dz \cdot dt = 0$;
- $dt^i = 0$ for $i > 1$.

Generalized Wiener processes

We can add a drift term and then define the generalized Wiener process:

$$dx = adt + bdz$$

with $dz = \epsilon dt$, $dz \sim N(0, dt)$ and $a, b \in \mathbb{R}$. adt is the drift term and bdz is the volatility term.

Then we obtain that the mean is $a\,dt$

$$E(dx) = adt + bE(dz) = adt$$

and the variation is given by the drift term

$$Var(dx) = b^2 Var(dz) = b^2 dt.$$

Hence, $dx \sim N(adt, b^2 dt)$.

1.1.2 Itô process

The *Itô* process is a generalized Wiener process where the parameters a and b are functions of the value of the variable, x and of the time t.

$$dx = a(x, t)dt + b(x, t)dz.$$

In this case the drift and the variance rate can change in time.

Itô lemma

The price of a stock option is a function of the underlying stock price and time. Suppose that the value of a variable x follows the *Itô* process

$$dx = a(x,t)dt + b(x,t)dz\,,$$

where dz is a Wiener process and a and b are functions of x and t.

Itô's lemma shows that a function G of x and t , infinitely differentiable, follows the process

$$\begin{aligned} dG(x,t) &= \frac{\partial G}{\partial x}dx + \frac{\partial G}{\partial t}dt + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}b^2 dt \\ &= \frac{\partial G}{\partial x}(adt + bdz) + \frac{\partial G}{\partial t}dt + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}b^2 dt \\ &= \left(a\frac{\partial G}{\partial x} + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}b^2\right)dt + b\frac{\partial G}{\partial x}dz \\ &= a^* dt + b^* dz \end{aligned}$$

Then also G follows a Itô process.

We can consider x, the stock price and G, the price of the derivative both functions of x. The volatility is all in dz as the underlying stock.

We can also examine the stochastic process followed by non-dividend-paying stock. We can not consider a constant drift term. If S is the stock price, we have

$$dS = \mu S dt + \sigma S dz$$

or

$$\frac{dS}{S} = \mu dt + \sigma dz.$$

which corresponds to the most widely used model of stock price behaviour. The variable σ is the volatility of the stock price, and the variable μ is the expected rate of return.

1.1.3 The Lognormal property

We use Itô's Lemma to derive a process followed by $\ln S$ when S follows the Itô's process.

We define $G = \ln S$ since

$$\begin{aligned} dG &= \frac{\partial G}{\partial S}dS + \frac{1}{2}\frac{\partial^2 G}{\partial S^2}(dS)^2 \\ &= \frac{1}{S}dS + \frac{1}{2}\left(-\frac{1}{S^2}\right)(dS)^2 \end{aligned}$$

Then

$$d(\ln S) = \mu dt + \sigma dz - \frac{1}{2}\sigma^2 dt$$
$$= \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma dz$$

This last equation indicates that $\ln S$ follows a generalized Wiener process. It has a constant drift $\mu - \sigma^2/2$ and constant variance rate σ^2.

The change in $\ln S$ is normally distributed as

$$\ln S \sim N(\left(\mu - \frac{1}{2}\sigma^2\right) T, \sigma\sqrt{T}).$$

1.1.4 Black-Scholes-Merton model

The partial differential equation of Black-Scholes-Merton is the equation that must be satisfied by the price of whatever financial derivative that depends on non dividend paying stock.

Derivation of the differential equation

The value of a call option is a function of various parameters, such as the strike price K and the time to maturity $T - t$, where T is the date of maturity and t is the current time. The value will also depend on properties of the asset itself, like its price, its drift and its volatility, as well as the risk-free rate if interest. Then we can write the option value as:

$$V = V(S, t; \sigma, \mu; K, T; r)$$

where

- S and t are variables;
- σ and μ are parameters associated with the asset price;
- K and T are parameters fixed in the contract;
- r is a parameter associated with the currency in which the asset is quoted.

We use $V(S, t)$ to denote the *option value*. Let π be the value of a portfolio of a long option position and a short position in a same Δ of underling. Its value is:

$$\pi = V(S, t) - \Delta\,. \tag{1.1}$$

We assume that the underling follows a Itô's process:

$$dS = \mu S dt + \sigma S dz. \tag{1.2}$$

The change in the portfolio's value from time t to $t + dt$ is partly due to the change in the option value and partly to the change in underlying:

$$d\pi = dV - \Delta dS. \tag{1.3}$$

Applying Itoô's lemma to the expression of V, we get

$$dV = \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S} dS + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} dt\,. \tag{1.4}$$

Then, the portfolio's variation is

$$d\pi = \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S} dS + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} dt - \Delta S\,. \tag{1.5}$$

The right-side of 1.5 contains deterministic terms and stochastic terms. The deterministic terms are those with dt, and the stochastic ones are those with dS. Then we know everything of the right-hand side except for the value of dS. These stochastic terms are the risk of portfolio. Eliminating the risk we must be careful to choose the Δ. In fact, the stochastic terms are:

$$\left(\frac{\partial V}{\partial S}\Delta\right) dS.$$

If we then choose:

$$\Delta = \frac{\partial V}{\partial S} \tag{1.6}$$

the risk is eliminated.

The reduction of risk to zero is called *hedging.* The perfect elimination of risk, by exploiting correlation between two tools, option and its underlying, is called *delta hedging.*

Delta hedging is an example of dynamic hedging strategy. From one timestep to the next the quantity $\frac{\partial V}{\partial S}$ changes, since it behaves like V, a function of S and t. This means that the perfect hedge must be continuosly rebalanced.

After choosing Δ the value of the portfolio is:

$$d\pi = \left(\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) dt. \tag{1.7}$$

This variation is completely riskless. If we have a completely risk-free change $d\pi$ in the portfolio value π then it must be the same as the growth we would get if we put the equivalent amount of cash in a risk-free securities

$$d\pi = r\pi dt\,. \tag{1.8}$$

This is the so-called *no arbitrage principle.* In fact, if the securities earn more than the portfolio, arbitrageurs could make a riskless profit by borrowing money to buy