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Preface

The aim of these lecture notes is to present new results, often in a review form, in order to provide an overview of the state-of-the-art research in the field. Most of these results were presented at the eighth Meeting on Stochastic Partial Differential Equations and Applications, held in Levico Terme in January 2008. This conference brought together a particularly distinguished and representative group of researchers in the field. Among the topics discussed:

1 Stochastic partial differential equations: general theory and applications.

2 Finite and infinite dimensional diffusion processes.

3 Stochastic calculus.

4 Theory of interacting particles.

5 Quantum probability.

6 Stochastic control.

The conference was financed by the CIRM (Centro Internazionale per la Ricerca Matematica) and INDAM (Istituto Nazionale di Alta Matematica "F. Severi").

The Organizing Committee (F. Flandoli, D. Nualart, E. Pardoux, M. Röckner and the editors) would like to express their warmest thanks to the Secretary of the CIRM, Augusto Micheletti, for his continuous assistance.

Giuseppe Da Prato and Luciano Tubaro

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Infinite dimensional integrals and their asymptotics: some recent developments and applications

S. Albeverio and S. Mazzucchi

Contents

1. Introduction

The study of the detailed asymptotic behavior in the limit $\epsilon \downarrow 0$ of infinite dimensional integrals of the form

$$\int_{\mathcal{B}} e^{\frac{F(\epsilon x)}{\epsilon^2}} G(\epsilon x)\mu(dx) \tag{1.1}$$

(where ϵ is a real positive parameter, μ is a Gaussian measure on a Banach space $\mathcal{B}$ and F, G are Borel measurable functionals on $\mathcal{B}$) by means of an infinite dimensional version of the Laplace method is a classical topic of investigation. First results were obtained by Donsker's school, in particular by Schilder [73], for the asymptotics of classical Wiener integrals, where B is the space of continuous functions with the sup norm and μ is the Wiener measure. Schilder's main theorems were generalized by Pincus [66] to the case of more general Gaussian functional integrals, by Kallianpur and Oodaira [57] to an abstract Wiener space setting, and by Ben Arous and Léandre [35, 36] to the case of path space measures associated to stochastic differential equations. These results were successfully applied to the study of the asymptotics (for small values of parameters, e.g time or the diffusion coefficient) of the solutions of some partial (stochastic) differential equations, see, e.g., [2, 58]. For some further rather recent extensions and results see e.g. [21, 31, 47, 48, 67, 68, 72, 29, 16, 62]. A related, but not identical problem, consists in studying the asymptotics for $\epsilon \downarrow 0$ of integrals of the form

$$\int_{\mathcal{B}} e^{-\frac{\Phi(x)}{\epsilon}} G(x)\mu(dx) \tag{1.2}$$

where μ is a smooth measure and Φ is a lower bounded "phase function", see [31, 30]. This is directly related to the classical Laplace method for finite dimensional integrals (see, e.g., [67]). [35, 30] also contains a study of asymptotics for solution of an infinite dimensional heat equation with potential.

Another interesting problem is the study of the detailed asymptotic behavior in the limit $\epsilon \downarrow 0$ of infinite dimensional oscillatory integrals of the form

$$\int_{\mathcal{H}} e^{-\frac{i}{\epsilon}\Phi(x)} G(x)dx \tag{1.3}$$

(where $\mathcal{H}$ is a real separable Hilbert space, Φ, G Borel measurable functionals on $\mathcal{H}$ satisfying suitable assumptions) by means of an infinite dimensional version of the stationary phase method. Integrals of the form (1.3) arise as "Feynman path integrals" in quantum mechanics, with ϵ being the Planck's constant, the asymptotics for $\epsilon \downarrow 0$ corresponding to "extracting classical mechanics from quantum mechanics". For the definition of the integral (1.3), its application to quantum mechanical problems as well as the detailed investigation of its limit when $\epsilon \downarrow 0$ see for instance [2, 18, 7, 35, 28, 8] and references therein.

The present paper presents some results concerning the asymptotics of infinite dimensional integrals of types (1.1), (1.2) and (1.3). Section 2 describes infinite dimensional oscillatory integrals of the type (1.3) with a polynomial phase function Φ connecting them with Gaussian integrals of the type (1.1) by means of an analytic continuation technique. Section 3 discusses briefly applications of integrals of the form (1.1) - (1.3) to some stochastic differential equations with applications to problems arising, e.g., in the theory of financial markets or in neurobiological systems.

2. Asymptotic expansion of infinite dimensional oscillatory integrals

Oscillatory integrals on finite dimensional Hilbert spaces, i.e. expressions of the form

$$\int_{\mathbb{R}^n} e^{-\frac{i}{\epsilon}\Phi(x)} g(x)dx, \tag{2.1}$$

(where $\Phi : \mathbb{R}^n \to \mathbb{R}$ is the phase function and $\epsilon \in \mathbb{R}^+$ a real positive parameter) and their limiting behaviour as $\epsilon \downarrow 0$ constitute a classical topic of investigation, having several applications, e.g. in electromagnetism, optics and acoustics. The proper interest of the study of (2.1) lies in the case where g is not summable and one has to exploit cancellations in (2.1) due to the wild oscillations of the integrand. The roots of the study of such integrals and their asymptotics lie in the classical method of stationary phase developed by Kelvin, Stokes and others in the 19th century. In the middle of last century it was developed systematically particularly by Hörmander [53], Maslov [63] and their coworkers, and became a basic instrument for the theory of Fourier integral operators and micro-local analysis.

If the phase function Φ is lower semibounded, it is also possible to consider the more general case where the imaginary unity i in the exponent of (2.1) is replaced by a complex parameter $s \in \mathbb{C}^+ \equiv \{z \in \mathbb{C} \,:\, \mathrm{Re}(z) \geq 0\}$:

$$I(s) \equiv \int_{\mathbb{R}^n} e^{-\frac{s}{\epsilon}\Phi(x)} g(x) dx. \tag{2.2}$$

In the case where the phase function Φ is a quadratic form, the integral (2.2) is called *Fresnel integral*. In [23, 24, 26] a modification of the Hörmander's definition [53] of oscillatory integrals (2.1) has been proposed, which is suitable to cover the general case (2.2), originally considered in [46, 7] in connection with the generalization to the infinite dimensional case. This modification is as follows:

Definition 2.1. *Let $f : \mathbb{R}^n \to \mathbb{C}$ be a Borel function, $s \in \mathbb{C}^+$ a complex parameter. Let $\mathcal{S}$ be a subset of the space of the Schwartz test functions $S(\mathbb{R}^n)$. If for each $\phi \in \mathcal{S}$ such that $\phi(0) = 1$ the integrals*

$$I_\delta(f, \phi) := \int_{\mathbb{R}^n} (2\pi s^{-1})^{-n/2} e^{-\frac{s}{2}|x|^2} f(x)\phi(\delta x) dx$$

exist for all $\delta > 0$ and $\lim_{\delta \to 0} I_\delta(f, \phi)$ exists and is independent of ϕ, then this limit is called the normalized Fresnel integral of f with parameter s (with respect to the space $\mathcal{S}$ of regularizing functions) and denoted by

$$\mathcal{F}^s(f) \equiv \widetilde{\int_{\mathbb{R}^n}^{s}} e^{-\frac{s}{2}|x|^2} f(x) dx. \tag{2.3}$$

We remark that s in definition 2.1 plays the role of s/ϵ in (2.2) and a suitable normalization has been introduced in order to be able later to pass to the limit $n \to \infty$. One shows then that (2.3) exists for f belonging to a large class of functions, containing in particular functions of the form $f(x) = e^{-\Phi(x)} g(x)$, with Φ, g of (at most) polynomial growth. See [23] for a detailed study of (2.3) and its asymptotics for $s = \tilde{s}/\epsilon$, $Re(\tilde{s}) > 0$, $\epsilon \downarrow 0$ (e.g. results on analyticity resp. Borel summability in ϵ, according to the assumptions on Φ resp. g).