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Editorial

This special issue of *Note di Matematica* contains the complete texts of the extended lectures course series presented as part of the *Advances Group Theory and Applications 2007* international conference held in Otranto from 4th to 8th, 2007.

The organising committee is expressly grateful to the authors for the range and detail of their scholarly contributions. These form a comprehensive summary and an invaluable basis for further research in the chosen areas. Further, they will provide stimulus and direction in the search for new important results. Special thanks are due also to the contributors to the wide programme of short talks and to all the partecipants for making the event such a resounding success.

A conference of this magnitude could not be organised without the generous support of many others. In this regard, we wish especially to express our deepest gratitude to *Dipartimento di Matematica e Applicazioni "R. Caccioppoli" of Napoli, Dipartimento di Matematica "E. De Giorgi" of Lecce* and *Ministero dell'Istruzione, dell'Università e della Ricerca.*

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Groups with all subgroups subnormal

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Abstract. An updated survey on the theory of groups with all subgroups subnormal, including a general introduction on locally nilpotent groups, full proofs of most results, and a review of the possible generalizations of the theory.

Keywords: Subnormal subgroups, locally nilpotent groups.

MSC 2000 classification: 20E15, 20F19

1 Locally nilpotent groups

In this chapter we review part of the basic theory of locally nilpotent groups. This will mainly serve to fix the notations and recall some definitions, together with some important results whose proofs will not be included in these notes. Also, we hope to provide some motivation for the study of groups with all subgroups subnormal (for short $\mathcal{N}_1$-groups) by setting them into a wider frame. In fact, we will perhaps include more material then what strictly needed to understand $\mathcal{N}_1$-groups.

Thus, the first sections of this chapter may be intended both as an unfaithful list of prerequisites and a quick reference: as such, most of the readers might well skip them. As said, we will not give those proofs that are too complicate or, conversely, may be found in any introductory text on groups which includes some infinite groups (e.g. [97] or [52], for nilpotent groups we may suggest, among many, [56]). For the theory of generalized nilpotent groups and that of subnormal subgroups, our standard references will be, respectively, D. Robinson's classical monography [96] and the book by Lennox and Stonehewer [64].

In the last section we begin the study of $\mathcal{N}_1$-groups, starting with the first basic facts, which are not difficult but are fundamental to understand the rest of these notes.

1.1 Commutators and related subgroups

Let x, y be elements of a group G. As customary, we denote by $x^y = y^{-1}xy$ the conjugate of x by y. The *commutator* of x and y is defined in the usual way as

$$[x, y] = x^{-1}y^{-1}xy = x^{-1}x^y.$$

Then, for $n \in \mathbb{N}$, the iterated commutator $[x,_n y]$ is recursively defined as follows

$$[x,_0 y] = x, \quad [x,_1 y] = [x, y]$$

and, for $1 \leq i \in \mathbb{N}$,

$$[x,_{i+1} y] = [[x,_i y], y].$$

Similarly, if $x_1, x_2, \dots x_n$ are elements of G, the *simple* commutator of weight n is defined recursively by

$$[x_1, x_2, \dots, x_n] = [[x_1, \dots, x_{n-1}], x_n].$$

We list some elementary but important facts of commutator manipulations. They all follow easily from the definitions, and can be found in any introductory text in group theory.

1 Lemma. *Let G be a group, and $x, y, z \in G$. Then*

(1) $[x, y]^{-1} = [y, x]$*;*

(2) $[xy, z] = [x, z]^y[y, z] = [x, z][x, z, y][y, z]$*;*

(3) $[x, yz] = [x, z][x, y]^z = [x, z][x, y][x, y, z]$*;*

(4) *(Hall-Witt identity)* $[x, y^{-1}, z]^y[y, z^{-1}, x]^z[z, x^{-1}, y]^x = 1.$

2 Lemma. *Let G be a group, $x, y \in G$, $n \in \mathbb{N}$, and suppose that $[x, y, y] = 1$; then $[x, y]^n = [x, y^n]$. If further $[x, y, x] = 1$, then*

$$(xy)^n = x^n y^n [y, x]^{\binom{n}{2}}.$$

If X is a subset of a group G then $\langle X \rangle$ denotes the subgroup generated by X. If U and V are non-empty subsets of the group G, we set

$$[U, V] = \langle [x, y] \mid x \in U, \ y \in V \rangle$$

and define inductively in the obvious way $[U,_n V]$, for $n \in \mathbb{N}$. Finally, if $A \leq G$, and $x \in G$, we let, for all $n \in \mathbb{N}$, $[A,_n x] = \langle [a,_n x] \mid a \in A \rangle$.

If $H \leq G$, H_G denotes the largest normal subgroup of G contained in H, and H^G the *normal closure* of H in G, i.e. the smallest normal subgroup of G containing H. Clearly,

$$H_G = \bigcap_{g \in G} H^g \qquad \text{and} \qquad H^G = \langle H^g \mid g \in G \rangle.$$

More generally, if X and Y are non-empty subsets of the group G, we denote by X^Y the subgroup $\langle x^y \mid x \in X,\ y \in Y \rangle$.

The following are easy consequences of the definitions.

3 Lemma. *Let H and K be subgroups of a group. Then $[H, K] \trianglelefteq \langle H, K \rangle$.*

4 Lemma. *Let X, Y be subsets of the group G. Then*

$$[\langle X \rangle, \langle Y \rangle] = [X, Y]^{\langle X \rangle \langle Y \rangle}.$$

If $N \trianglelefteq G$, then $[N, \langle X \rangle] = [N, X]$.

The next, very useful Lemma follows from the Hall-Witt identity.

5 Lemma. [Three Subgroup Lemma]. *Let A, B, C be subgroups of the group G, and let N be a normal subgroup such that $[A, B, C]$ and $[B, C, A]$ are contained in N. Then also $[C, A, B]$ is contained in N.*

The rules in Lemma 1, as well as others derived from those, may be applied to get sorts of handy analogues for subgroups. For instance, if A, B, C are subgroups of G and $[A, C]$ is a normal subgroup, then $[AB, C] = [A, C][B, C]$. More generally, we have

6 Lemma. *Let N, $H_1, \ldots, H_n$ be subgroups of the group G, with $N \trianglelefteq G$, and put $Y = \langle H_1, \ldots, H_n \rangle$. Then*

$$[N, Y] = [N, H_1] \ldots [N, H_n].$$

The same commutator notation we adopt for groups actions: let the group G act on the group A. For all $g \in G$ and $a \in A$, we set $[a, g] = a^{-1}a^g$, and $[A, G] = \langle [a, g] \mid a \in A, g \in G \rangle$. With the obvious interpretations, the properties listed above for standard group commutators continue to hold.

For a group G, the subgroup $G' = [G, G]$ is called the *derived* subgroup of G, and is the smallest normal subgroup N of G such that the quotient G/N is abelian. The terms $G^{(d)}$ $(1 \leq d \in \mathbb{N})$ of the *derived series* of G are the characteristic subgroups defined by $G^{(1)} = G'$ and, inductively ,

$$G^{(n+1)} = (G^{(n)})' = [G^{(n)}, G^{(n)}]$$

(the second derived subgroup $G^{(2)}$ is often denote by G''). The group G is *soluble* if there exists an n such that $G^{(n)} = 1$; in such a case the smallest integer n for which this occurs is called the *derived length* of the soluble group G. Of course, subgroups and homomorphic images of a soluble group of derived length d are soluble with derived length at most d.

A group is said to be *perfect* if it has no non-trivial abelian quotients; thus, G is perfect if and only if $G = G'$.

By means of commutators are also defined the terms $\gamma_d(G)$ of the *lower central series* of a group G: set $\gamma_1(G) = G$, and inductively, for $d \geq 1$,

$$\gamma_{d+1}(G) = [\gamma_d(G), G] = [G,_d G].$$

These are also characteristic subgroups of G. A group G is *nilpotent* if, for some $c \in \mathbb{N}$, $\gamma_{c+1}(G) = 1$. The *nilpotency class* (or, simply, the class) of a nilpotent group G is the smallest integer c such that $\gamma_{c+1}(G) = 1$.

7 Lemma. *Let G be a group, and $m, n \in \mathbb{N} \setminus \{0\}$. Then*

(1) $[\gamma_n(G), \gamma_m(G)] \leq \gamma_{n+m}(G)$;.

(2) $\gamma_m(\gamma_n(G)) \leq \gamma_{mn}(G)$;

From (1), and induction on n, we have

8 Corollary. *For any group G and any $1 \leq n \in \mathbb{N}$, $G^{(n)} \leq \gamma_{2^n}(G)$. In particular a nilpotent group of class c has derived length at most $[\log_2 c] + 1$.*

Also, by using (1) and induction, one easily proves the first point of the following Lemma, while the second one follows by induction and use of the commutator identities of 1,

9 Lemma. *Let G be a group, and $1 \leq n \in \mathbb{N}$. Then*

(1) $\gamma_n(G) = \langle [g_1, g_2, \ldots, g_n] \mid g_i \in G,\ i = 1, 2, \ldots, n \rangle$.

(2) *If S is a generating set for G, then $\gamma_n(G)$ is generated by the simple commutators of weight at least n in the elements of $S \cup S^{-1}$.*

The *upper central series* of a group G is the series whose terms $\zeta_i(G)$ are defined in the familiar way: $\zeta_1(G) = Z(G) = \{x \in G \mid xg = gx\ \forall g \in G\}$ is the centre of G, and for all $n \geq 2$, $\zeta_n(G)$ is defined by

$$\zeta_n(G)/\zeta_{n-1}(G) = Z(G/\zeta_{n-1}(G)).$$

A basic observation is that, for $n \geq 1$, $\zeta_n(G) = G$ if and only if $\gamma_{n+1}(G) = 1$, and so G is nilpotent of class c if and only if $G = \zeta_c(G)$ and c is the smallest such positive integer. This follows at once from the following property.

10 Lemma. *Let G be a group, and $1 \leq n \in \mathbb{N}$. Then $[\gamma_n(G), \zeta_n(G)] = 1$.*

The next remark is often referred to as *Grün's Lemma.*

11 Lemma. *Let G be a group. If $\zeta_2(G) > \zeta_1(G)$ then $G' < G$.*

Let us recall here some elementary but more technical facts, which we will frequently use, about commutators in actions on an abelian groups.

Thus, let A be a normal abelian subgroup of a group G, $F \leq A$, and let $x \in G$. It is then easy to see that, for all $i \in \mathbb{N}$,

$$[F,_i x] = \{ \ [a,_i x] \mid a \in F \ \} \quad \text{and} \quad F^{\langle x \rangle} = \langle \ [F,_i x] \mid i \in \mathbb{N} \ \rangle.$$

12 Lemma. *Let A be a normal abelian subgroup of the group G, and $H \leq G$. Suppose that $H/C_H(A)$ is abelian. Then, for all $a \in A$, $x, y \in H$:*

$$[a, x, y] = [a, y, x].$$

Proof. Since $H/C_H(A)$ is abelian, $[a, xy] = [a, yx]$, and, by expanding the commutators using Lemma 1, $[a, y][a, x]^y = [a, x][a, y]^x$. Since A is abelian, we get the desired equality $[a, x]^{-1}[a, x]^y = [a, y]^{-1}[a, y]^x$. QED

13 Corollary. *Let A be a normal abelian subgroup of the group G, such that $G/C_G(A)$ is abelian. Then, for all $X, Y \leq G$: $[A, X, Y] = [A, Y, X]$.*

14 Lemma. *Let A be a a normal elementary abelian p-subgroup of a group G. Then, for all $x \in G$, $[A,_{p^m} x] = [A, x^{p^m}]$ for all $m \in \mathbb{N}$.*

Proof. It is convenient to look at x as to an endomorphism, via conjugation, of the abelian group A. Then, for all $a \in A$, $[a, x] = a^{-1}a^x = a^{x-1}$, whence, as A has exponent p,

$$[a,_p x] = a^{(x-1)^p} = a^{x^p - 1} = [a, x^p]$$

and the inductive extension to any power x^{p^m} is immediate. QED

15 Corollary. *Let $1 \neq A$ be a normal elementary abelian p-subgroup of the group G. If $G/C_G(A)$ is a finite p-group, then there exists $n \geq 1$ such that $A \leq \zeta_n(G)$.*

Proof. Let $C = C_G(A)$. We argue by induction on m, where $|G/C| = p^m$. If $m = 0$, A is central in G. Thus, let $m \geq 1$, N/C a maximal subgroup of G/C, and $x \in G \setminus N$. Then, by inductive assumption, $A \leq \zeta_k(N)$, for some $k \geq 1$. Let $A_0 = \zeta(N) \cap A$; then $A_0 \neq 1$ and $C_G(A_0) \geq N$. Now, $x^p \in N$, and by Lemma 14

$$[A_0,_p x] = [A_0, x^p] \leq [A_0, N] = 1.$$

This means that $A_0 \leq \zeta_p(G)$. Ny repeating this same argument for all the central N-factors contained in A, we get $[A,_{pk} G] = 1$, whence $A \leq \zeta_{pk}(G)$. QED

16 Lemma. *Let A be an abelian group, and x an automorphism of A such that $[A,_n x] = 1$, for $n \geq 1$.*

(i) *If x has finite order q, then $[A^{q^{n-1}}, x] = 1$.*

(ii) *If A has finite exponent $e \geq 2$, then $[A, x^{e^{n-1}}] = 1$.*

(iii) *Let the group H act on A with $[A,_n H] = 1$ ($n \geq 1$); then $\gamma_n(H) \leq C_H(A)$.*

Proof. (i) By induction on n. If $n = 1$ we have nothing to prove. Thus, let $n \geq 2$, and set $B = [A, x]$. Then $[B,_{n-1} x] = 1$, whence, by inductive assumption,

$$[A^{q^{n-2}}, x, x] = [[A, x]^{q^{n-2}}, x] = [B^{q^{n-2}}, x] = 1.$$

Now, let $b \in A^{q^{n-2}}$. Then, since $[b, x, x] = 1 = [b, x, b]$, by Lemma 2 we have $[b^q, x] = [b, x]^q = [b, x^q] = 1$. Hence, $[A^{q^{n-1}}, x] = [(A^{q^{n-2}})^q, x] = 1$, as wanted.

(ii) By induction on n. If $n = 1$, then $1 = [A, x] = [A, x^{e^0}]$. Let $n \geq 2$, and set $B = [A, x^{e^{n-2}}] \leq [A, x]$. Then, by inductive hypothesis,

$$[A, x^{e^{n-2}}, x^{e^{n-2}}] = [B, x^{e^{n-2}}] = 1 \ .$$

By Lemma 2, we then have $[A, x^{e^{n-1}}] = [A, x^{e^{n-2}e}] = [A, x^{e^{n-2}}]^e = 1$.

(iii) By induction on n, being the case $n = 1$ trivial. Let $n > 1$. Then H acts on $[A, H]$ and $[A, H,_{n-1} H] = 1$, hence, by inductive assumption

$$[A, H, \gamma_{n-1}(H)] = 1. \tag{1}$$

Let $A_0 = [A,_{n-1} H]$ and $\overline{A} = A/A_0$. Then H acts on $\overline{A}$ and $[\overline{A},_{n-1} H] = 1$. By inductive assumption we have $[\overline{A}, \gamma_{n-1}(H)] = 1$, which means $[\gamma_{n-1}, A] \leq A_0$. Since $[A_0, H] = 1$, we get $[\gamma_{n-1}(H), A, H] = 1$, which, together with (1 and the Three Subgroup Lemma, yields $\gamma_n(H), A] = [H, \gamma_{n-1}(H), A] = 1$. QED

Point (iii) of Lemma 16 is a particular case of a theorem of Kalužnin, which we will state later, together with an important generalization due to P. Hall.

It is not difficult to extend similar remarks to the case when A is nilpotent. in which case it is to be expected that the numerical values will depend also on the nilpotency class of A. We show only one of these possible generalizations.

17 Lemma. *Let A be a nilpotent group of class c, and x an automorphism of A such that $|x| = q$ and $[A,_n x] = 1$, for $n \geq 1$. Then $[A^{q^{cn-1}}, x] = 1$.*

Proof. We argue by induction on the class c of A. The case $c = 1$ is just point (i) of the previous Lemma. Thus, we assume $c \geq 2$ and write $B = A^{q^{(c-1)n-1}}$. Then, by inductive assumption,

$$[B, x] \leq \gamma_c(A) \leq Z(A).$$

In particular, $[B, x, B] = 1$, and so by Lemma 2, $[B^{q^{n-1}}, x] = [B, x]^{q^{n-1}}$. Also, $[B, x]$ is abelian and so $[[B, x], x] = [B, x, x]$. Thus, by case $c = 1$, $[[B, x]^{q^{n-1}}, x] = 1$. Hence $[B^{q^{n-1}}, x, x] = 1$. Thus

$$[B^{q^n}, x] = [B^{q^{n-1}}, x]^q = [B^{q^{n-1}}, x^q] = 1.$$

Therefore, $A^{q^{cn-1}} = B^{q^n} \leq C_A(x)$, as wanted. QED

Let us state a handy corollary, for which we need to fix the following notation. Given a group G, and an integer $n \geq 1$, we denote by G^n the subgroup of G generated by the n-th powers of all the elements of G, and set $G^\omega = \bigcap_{n\in\mathbb{N}} G^n$.

18 Corollary. *Let G be a periodic nilpotent group. Then $G^\omega \leq Z(G)$.*

Now a technical result (Lemma 21) which will be very useful. For the proof we first need the following observation

19 Lemma. *Let A be a nilpotent group of class $c > 0$, and let x be an automorphism of A. Then, for every $q \geq 1$,*

$$[A^{q^c}, \langle x\rangle] \leq [A, \langle x\rangle]^q.$$

Proof. By induction on c. If $c = 1$ we have equality $[A^q, \langle x\rangle] = [A, \langle x\rangle]^q$. Thus, let $c \geq 2$, $T = \gamma_c(A)$, and set $D = [A, \langle x\rangle]^q$. Then, D is normal in A and $\langle x\rangle$-invariant. By inductive assumption, $[A^{q^{c-1}}, \langle x\rangle] \leq DT$; i.e., setting $\overline{A} = A/D$,

$$[\overline{A}^{q^{c-1}}, \langle x\rangle] \leq \overline{T} \leq Z(\overline{A}).$$

If $a \in A$ and $u = a^{q^{c-1}}$, we have $[Du, \langle x\rangle] \leq \overline{T}$, and so $[Du^q, x] = [Du, x]^q = 1$, which is to say that

$$[a^{q^c}, \langle x\rangle] = [u^q, \langle x\rangle] \subseteq D = [A, \langle x\rangle]^q,$$

thus completing the proof. QED

20 Corollary. *Let A be a nilpotent group of class $c > 0$, and let $x_1, \dots, x_d$ be automorphisms of A. Then, for every $q \geq 1$,*

$$[A^{q^{c^d}}, \langle x_1\rangle, \dots \langle x_d\rangle] \leq [A, \langle x_1\rangle, \dots, \langle x_d\rangle]^q.$$

21 Lemma. *Let A be a nilpotent group of class c, let $x_1, x_2, \ldots, x_d$ be automorphisms of A such that $[A,_n \langle x_i \rangle] = 1$ for all $i = 1, \ldots, d$. Let $q_1, \ldots, q_d$ be integers ≥ 1, and $q = q_1 \cdots q_d$. Then*

$$[A^{q^{nc^d}}, \langle x_1 \rangle, \ldots, \langle x_d \rangle] \leq [A, \langle x_1^{q_1} \rangle, \ldots, \langle x_d^{q_d} \rangle].$$

Proof. We argue by induction on $d \geq 1$. If $d = 1$, $q = q_1$, write $R = [A, \langle x^q \rangle]$. Then $R \trianglelefteq \langle A, x \rangle$, and by applying Lemma 17 to the action of x on A/R, we have (since x^q centralizes A/R)

$$[A^{q^{cn}}, \langle x \rangle] \leq R$$

which is what we want.

Let then $d \geq 2$. Write $s = q_1 \ldots q_{d-1}$ and $B = [A^{s^{nc^{d-1}}}, \langle x_1 \rangle, \ldots, \langle x_{d-1} \rangle]$. By inductive assumption

$$B \leq [A, \langle x_1^{q_1} \rangle, \ldots, \langle x_{d-1}^{q_{d-1}} \rangle]. \tag{2}$$

Now, $q^{nc^d} = s^{nc^d} q_d^{nc^d}$; thus, using Corollary 20,

$$[A^{q^{nc^d}}, \langle x_1 \rangle, \ldots, \langle x_d \rangle] \leq [[A^{s^{nc^d}}, \langle x_1 \rangle, \ldots, \langle x_{d-1} \rangle]^{q_d^{nc}}, \langle x_d \rangle] \leq [B^{q_d^{nc}}, \langle x \rangle].$$

By the case $d = 1$ we then have

$$[A^{q^{nc^d}}, \langle x_1 \rangle, \ldots, \langle x_d \rangle] \leq [B^{\langle x_q \rangle}, \langle x_d^{q_d} \rangle] = [B, \langle x_d^{q_d} \rangle],$$

from which, applying (2), we get the desidered inclusion. QED

1.2 Subnormal subgroups and generalizations

A subgroup H of the group G is said to be *subnormal* (written $H \triangleleft\triangleleft G$) if H is a term of a finite series of G; i.e. if there exists $d \in \mathbb{N}$ and a series of subgroups, such that

$$H = H_d \trianglelefteq H_{d-1} \trianglelefteq \ldots \trianglelefteq H_0 = G.$$

If $H \triangleleft\triangleleft G$, then the *defect* of H in G is the shortest lenght of such a series; it will be denoted by $d(H, G)$. We shall say that a subgroup H of G is *n-subnormal* if $H \triangleleft\triangleleft G$ and $d(H, G) \leq n$.

Clearly, subnormality is a transitive relation, in the sense that if $S \triangleleft\triangleleft H$ and $H \triangleleft\triangleleft G$, then $S \triangleleft\triangleleft G$. Moreover, if $S \triangleleft\triangleleft G$, then $S \cap H \triangleleft\triangleleft H$ for every $H \leq G$, and $SN/N \triangleleft\triangleleft G/N$ for every $N \trianglelefteq G$. Also, the intersection of a finite set of subnormal subgroups is subnormal; but this is not in general true for the intersection of an infinite family of subnormal subgroups. The join $\langle S_1, S_2 \rangle$ of two subnormal subgroups S_1 and S_2 is not in general a subnormal subgroup (see [64] for a full discussion of this point).

The reason why groups with all subgroups subnormal became a subject of investigation lies in the following elementary facts.

22 Proposition. *(1) In a nilpotent group of class c every subgroup is subnormal of defect at most c.*

(2) A finitely generated group in which every subgroup is subnormal is nilpotent.

Let $H \leq G$; the *normal closure series* $(H^{G,n})_{n\in\mathbb{N}}$ of H in G is defined recursively by

$$H^{G,0} = G, \quad H^{G,1} = H^G, \quad \text{and} \quad H^{G,n+1} = H^{H^{G,n}}.$$

By definition, $H^{G,n+1} \trianglelefteq H^{G,n}$, and it is immediate to show that if $H \triangleleft\triangleleft G$ and $H = H_d \trianglelefteq H_{d-1} \trianglelefteq \ldots \trianglelefteq H_0 = G$ is a series from H to G, then, for all $0 \leq n \leq d$, $H^{G,n} \leq H_n$. Thus, a subgroup H is subnormal in G if and only if $H^{G,d} \leq H$ for some $d \geq 0$, and the small such d is the defect of H. The following is easily proved by induction on n.

23 Lemma. *Let G be a group, and $H \leq G$. Then*

(1) *$H^{G,n} = H[G,_n H]$ for all $n \in \mathbb{N}$.*

(2) *For $d \geq 1$, H is d-subnormal if and only if $[G,_d H] \leq H$.*

For our purposes it is convenient to explicitely state also the following easy observation.

24 Lemma. *Let H be a subgroup of the group G and suppose that, for some $n \geq 1$, $H^{G,n} \neq H$. Then there exist finitely generated subgroups G_0 and H_0 of G and H, respectively, such that $[G_0,_n H_0] \not\leq H$.*

We recall another elementary and useful fact (for a proof see [64]).

25 Lemma. *Let H and K be subnormal subgroups of the group G. If $\langle H, K\rangle = HK$, then $\langle H, K\rangle$ is subnormal in G.*

Series. Although we will not be directly interested in generalizations of subnormality, we will sometimes refer to them, notably to ascendancy; also, when working with subnormal subgroups in infinite groups, in order to have a better understanding of what is going on, or to think to feasible extensions of our results, it may be useful to be aware of them.

Our definition of a (general) subgroup series in a group is the standard one proposed by P. Hall (which in turn includes the earlier Mal'cev's definition). We give only a brief resume of the principal features of this basic notion, by essentially reproducing part of §1.2 of [96], to which we refer for a fuller account.

Let Γ be a totally ordered set; a **series** of type Γ of a group G is a set

$$\{(V_\gamma, \Lambda_\gamma) \mid \gamma \in \Gamma\}$$

of pair of subgroups V_γ, Λ_γ of G such that

(i) $V_\gamma \trianglelefteq \Lambda_\gamma$ for all $\gamma \in \Gamma$;

(ii) $\Lambda_\alpha \leq V_\beta$ for all $\alpha < \beta$ $(\alpha, \beta \in \Gamma)$;

(iii) $G \setminus \{1\} = \bigcup_{\gamma\in\Gamma}(\Lambda_\gamma \setminus V_\gamma)$.

Each $1 \neq x \in G$ lies in one and only one of the difference sets $\Lambda_\gamma \setminus V_\gamma$. Moreover, for each $\gamma \in \Gamma$,

$$V_\gamma = \bigcup_{\beta<\gamma} \Lambda_\beta \qquad \Lambda_\gamma = \bigcap_{\beta>\gamma} V_\beta \tag{3}$$

unless γ is the least element (if it exists) of Γ, in which case $V_\gamma = 1$, or the greatest element, for which $\Lambda_\gamma = G$. The subgroups V_γ, Λ_γ are called the *terms* of the series, and the quotient groups Λ_γ/V_γ the *factors* of the series.

A series of a group G is called *normal* if every term is a normal subgroup of G, and *central* if every factor is a central factor of G (i.e. $[\Lambda_\gamma, G] \leq V_\gamma$ for all $\gamma \in \Gamma$). Clearly, every central series is also a normal series.

Let $\mathcal{S}$ and $\mathcal{S}'$ be two series of the same group G. We say that $\mathcal{S}'$ is a *refinement* of $\mathcal{S}$ if every term of $\mathcal{S}$ is also a term of $\mathcal{S}'$. This relation clearly defines a partial order relation on the set of all series of the group G, which it is easily seen to satisfy the chain condition, in the sense that every chain of series of G (with respect to the refinement relation) admits an upper bound. Thus, we may apply Zorn's Lemma to the set of all series of G to get series that are not refinable. These unrefinable series of G are called *composition series.* Thus,

26 Proposition. *For every series $\mathcal{S}$ of the group G there exists a composition series which is a refinement of $\mathcal{S}$.*

Clearly, a series $\mathcal{S}$ of G is a composition series if and only if all factors of $\mathcal{S}$ are non-trivial simple groups. If we restrict attention to normal series of G (or, more generally, to series all of whose terms are invariant under the action of a given operator group A), we can still apply Zorn's Lemma, and obtain maximal, that is unrefinable, normal series (or A-invariant series) of G; these are called *chief series*, or principal series, of G, and their factors are *chief factors* of G. Every group G admits composition series and chief series, but there is no analogue of the Jordan-Holder Theorem for finite groups (even the infinite cyclic group violates it).