Theory and Applications of Proximity, Nearness and Uniformity

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Editorial

Somashekhar (Som) Naimpally

Contents

1. Preface

> *"The significant problems we face cannot be solved by the same level of thinking that created them."*
>
> Albert Einstein

The main purpose of this preface is to give a short summary of the major developments of proximity in dealing with problems in topology and analysis. Naturally it reflects the bias and limitations of this author. Interested readers may find more details in the book "*Proximity Approach to Problems in Topology and Analysis*", Oldenbourg (2009) and in the references given at the end of this preface.

Intuitive Introduction

In this section, we propose to give a brief intuitive introduction to topology, proximity, and nearness. It shows that the concept of continuity and its various modifications can be explained even to persons who are not mathematicians.

It is not widely known to the public that abstract pure mathematics has applications in daily life. There are many examples in mathematics but here we wish to explain what is Topology and its applications. We use simple examples from day-to-day life to illustrate the concepts. Topology deals with the concept of *nearness* at various levels. This approach was first formulated by the Hungarian mathematician F. Riesz in an address to the International Congress of Mathematicians in Rome (1908).

Level 1 - Consider a typical family {Mother, Father, Son, Daughter}. We say that a person is near the family if that person is blood related to some member of the family. Of course, every member of the family is near the family and the family must first exist to talk about nearness! Grandparents, aunts, uncles, cousins etc. are persons near the family though they are not in the family. There are many other ways of defining such nearness relations, e.g. one may say that a person is near the family if the person helps the family in some way. In this definition the family physician, the plumber, the mailman etc. are near the family. This concept is axiomatized with a few simple obvious conditions and one gets the abstract concept of a topological space. This was accomplished by the Polish mathematician K. Kuratowski in 1922.

With a topological space, is associated another concept that of a *continuous* transformation. Suppose five years ago P was a family physician of the family F. That is the Physician P was near the family F under the rule that P helps the family in some way. Today, after five years, both the Physician P and the family F have changed. If the Physician P is still near the family F, we say that it is a continuous relationship in day-to-day life and the same is said in mathematics. If for any reason, the Physician P ceases to be the family doctor of F, we say that the relationship is discontinuous. *Thus a continuous transformation is one in which the nearness of a point to a set remains unchanged under that transformation.*

To recapitulate, in topology, we have nearness relations between points and sets, together with continuous transformations which preserve these nearness relations.

LEVEL 2 - At this level we talk about nearness between two families, technically called *proximity*. This idea, already present in Riesz's work, was thoroughly studied by the Soviet mathematician V. Efremovič around 1940 and published in 1951. This idea was further developed by the Soviet mathematician Yu. Smirnov. Again, this nearness between two families, can occur in several ways: (a) two families can be near because a daughter from one family has married a son from another, or (b) two families have a common friend (i.e. a person near both families), or (c) two families are interested in music and meet at a concert thus getting near each other. Moreover, in this subject one studies transformations which preserve nearness between pairs of sets. These are called *proximally continuous transformations*.

To recapitulate, in proximity, we have nearness relations between pairs of sets, together with proximally continuous transformations which preserve these nearness relations.

LEVEL 3 - Here one talks of nearness of a number of families technically resulting in a generalization of a uniform space which was discovered by the French mathematician A. Weil in 1937. An example is that of the families of persons who work for the same company. These families get together for a picnic or a Christmas party. Again, one has transformations which preserve this nearness of families and these are called *near transformations* or uniformly continuous transformations. In the most general sense, this was first studied by the German mathematician Horst Herrlich and this author in early 1970s.

It easy to see that the subject is international and mathematicians from all over the world have worked on this topic. Mathematicians work on this topic because the problems are interesting, challenging, or beautiful! Sometimes problems come from other areas but there are many instances where applications were discovered later. Abstract topology has found applications in Theoretical Computing, Quantum Mechanics, General Relativity, Mathematical Economics, Optimization, Convex Analysis, Probability Theory, Theory of Capacities, Child Psychology, a Model of our eyes, etc.
(*Abstract of a public lecture given by the author in 2004 in the conference Subtle Technologies in Toronto.*)

Teachers of calculus know how difficult it is to teach continuity in the classroom; see for example, Devlin's article on the website of the Mathematical Association of America [31]. For teaching calculus and advanced calculus using nearness see the article [26] or the booklet [87]. For teaching elementary general topology via nearness see [126].

Basics

> *"The existence of analogies between the central features of various theories implies the existence of a general theory which underlies the*

> *particular theories and unifies them with respect to these central features."*
>
> E. H. Moore

A topological space is sometimes defined in terms of the Kuratowski closure axioms (1922) [94]. If the axioms are written in the form of "a point is near a set", rather than in terms of the closure operator then they provide a motivation for the axioms for proximity: "one set is near another." This was done by Leader [103] for the non-symmetric proximities and by his student Lodato [110] for the symmetric ones in 1964. Proximity is a generalization of intersection of two sets which is replaced by nearness of two sets. Proximity is a finer structure than topology and provides a simple conceptual approach to many significant topological problems.

Since ancient times, mathematicians have struggled with the precise formulation of *continuity*. An abstract, yet a simple formulation, was given by F. Riesz in 1908 [172], using the concept of *nearness.*
Nearness is one of the rare concepts in the whole of mathematics that is at once intuitive and which can be made rigorous with little effort.
Since one uses the words *near* and *far* in day-to-day life, the concept of nearness is, even for non-mathematicians, easy to understand. It can be explained to "the first person one meets in the street", as the great mathematician Joseph Louis Lagrange said. And it can be made rigorous by formulating precise axioms. Its simplicity and depth provide a powerful tool in understanding many concepts as well as conducting research in topology and analysis.

Thus Topology is a subject in which, given a nonempty set X, one studies a nearness relation which determines whether or not a point x in X is *near* a subset B of X. If there is another nonempty set Y with a similar nearness relation, then a function $f : X \to Y$ is continuous if it preserves the nearness of points to subsets. As explained earlier, this can be easily related to the use of the term "continuous" in daily life and so one can explain *continuity* even to non-mathematicians. The category TOP consists of topological spaces as objects and continuous functions as morphisms.

Other possibilities include nearness of

(a) pairs of subsets (Proximity) [67];

(b) finitely many subsets (Contiguity) [92, 93];

(c) an arbitrary family of subsets (Nearness) [86].

When there are two spaces with structures of the same kind, functions preserving proximity (res. contiguity, nearness) are called proximally continuous (res. contigual, near) maps. Obviously we get the *categories* PROX, CONT and NEAR. For more information on categorical implications see [86].

Topology, proximity, contiguity, nearness give progressively finer structures. As per Einstein's quotation given above, the finer structures are useful in dealing with finer topological problems. Thus, the problem of finding necessary and sufficient conditions for the existence of continuous extensions of continuous functions from dense subspaces of Tychonoff spaces is purely topological. The problem has a simple solution in terms of proximity. Taimanov first proved a special case in which the range space is compact Hausdorff. His proof does not use proximity. But even a glance at the statement of his result reveals the hidden proximity that is obvious to those familiar with proximity! If the spaces involved are T_1, then one needs contiguity. If there is a need to compare sizes of neighbourhoods at various points, then one needs uniformity, which is essentially a special kind of nearness.

Extensions of spaces, compactifications

Set theoretic ultrafilters, their topological cousins closed or open ultrafilters and the uniform cousins Cauchy ultrafilters play important roles in topology in the study of many concepts such as compactness, H-closedness, completeness etc. They are also used in the construction of compactifications, H-closed extensions, completions etc. Now we study their analogues viz. clusters and bunches in proximity spaces first considered respectively by Leader [99] and Lodato [110]. Essentially, the condition "nonempty intersection" in ultrafilters is replaced by "near" in clusters or bunches. Moreover, as it turns out, bunches and clusters are unions of ultrafilters called grills first studied by Choquet [27] and later by Thron [199] in the context of proximity spaces. In an ultrafilter, union of two sets belongs to it iff one of the sets is in it. Proximity satisfies the union axiom and so the above condition is the basis for defining clusters and bunches.

Weierstrass, Dedekind and Cantor constructed the space of real numbers by the completion of the rational numbers. Since then mathematicians have constructed various extensions of a given topological space (X, T) lacking a property such as compactness, completeness, H-closedness etc. This was done by embedding (X, T), which lacks a certain property, into a space which has that property. In particular, there are several compactifications such as Alexandroff, Freudenthal, Stone-Čech, Wallman, ..., each having its own construction and serving its own purpose. The author agrees with E. H. Moore's statement quoted above. With the help of (Leader-Lodato) proximity and the tool of maximal bunches, a general T_1 compactification can be constructed which includes the various compactfications as special cases [77]. Other extensions can be constructed using Efremovič proximity and/or nearness. The method used is a modification of the one used by Wallman [204]. Wallman took the family $\mathrm{w}X$ of *all* closed ultrafilters of a T_1 space (X, T). For each closed set A in X, let A^* denote the family of members of $\mathrm{w}X$ which contain A. With the family $\{A^* : A \text{ closed in } X\}$ as a base for closed subsets, $\mathrm{w}X$ becomes a compact T_1 space in which X is embedded via the map which takes the point x in X to the closed ultrafilter of X which contains $\{x\}$. Since Wallman took *all* closed sets, this method yields only *one* compactification. By considering subfamilies of the family of closed subsets of X, and adding some conditions, Frink [76] was able to get many but not all Hausdorff compactifications. To get a general compactification, one has to modify the Wallman's original method and follow Leader and Lodato by assigning a compatible proximity to a T_1 space X and choose the family of all maximal bunches as the elements of the extension space. With the Wallman topology on this family and one gets a general T_1 compactification. In Tychonoff spaces with Efremovič proximity, the same construction yields all Hausdorff compactifications as Smirnov first showed.

The following table gives a detailed comparison between the Wallman compactification and the general compactification known as GN-compactification in the literature.