Recent Developments on Mathematical Programming and Applications

Workshop held in Pisa, on 5th June 2009

Edited by
Laura Carosi and Laura Martein
Contents

Efficiency under pseudoinvexity, and duality in differentiable and non-differentiable multiobjective problems. A characterization
Arana-Jiménez, Manuel ............................................... 1

Algorithmic aspects of a solution method for two low-rank generalized linear programs
Riccardo Cambini, Siegfried Schaible, Claudio Sodini ................. 15

Dual Pairs and Maximality
Erdo Castagnoli, Gino Favero ....................................... 27

On paramonotone and pseudomonotone maps
Marco Castellani, Massimiliano Giuli ................................ 41

Preinvex functions: some properties and comparisons
Giorgio Giorgi .................................................................. 57

Second order tangent sets
Laura Martein, Mauro Sodini ......................................... 67

Interval monotonicity and risk measures
Piera Mazzoleni .............................................................. 79

An interior point method for linearly constrained multiobjective optimization based on suitable descent directions
Enrico Miglierina, Elena Molho, Maria Cristina Recchioni .......... 89

A stochastic approach for Hierarchical Fleet Mix Problems
Rossana Riccardi .......................................................... 103
Preface

For several years, some members of both the Department of Quantitative Methods - University of Brescia and the Department of Statistics and applied Mathematics - University of Pisa, have been working together as members of the research project titled “Generalized Monotone Maps: theoretical aspects and solution algorithms”. The project, coordinated by Elisabetta Allevi (University of Brescia), was formed by the local research unit of Brescia and the local research unit of Pisa and it has been financially supported by MIUR, as a PRIN program.

The unit of Pisa organized the workshop “Recent developments on mathematical programming and applications”, as a closing activity of the project. The workshop took place at the Faculty of Economics of Pisa on 2009, 5th June with the purpose of consolidating the effective relationship between the two groups and permitting a profitable exchange of ideas among the members of the whole scientific community. Therefore, this one-day-conference has been attended both by members of the two research units and by scholars, whose research interests are very close to the topics of the PRIN project.

The present volume collects unpublished contributions related to some of the talks given at the workshop. The reader can find current theoretical studies on generalized convexity and generalized monotonicity. Both scalar and vector optimization problems are considered even from an algorithmic point of view and applications to finance and logistics are presented.

We wish to thank all those invited for coming and making the meeting so attractive, and all those who contributed to the realization of the present volume. Special thanks go to the MIUR for its essential financial support and the Faculty of Economics of Pisa for allowing us to organize the workshop inside the faculty.

Pisa, September 2009

Laura Carosi
Laura Martein
List of Contributors

**Arana Jiménez Manuel**
Departamento de Estadística e I.O.
Escuela Superior de Ingeniería
C/Chile SN 11002
Universidad de Cádiz
Despacho 2 planta Ed. Serv. Generales, Spain
manuel.arana@uca.es

**Cambini Riccardo**
Dipartimento di Statistica e Matematica applicata all’Economia
Università di Pisa
Via Ridolfi, 10 56124 Pisa, Italy
cambric@ec.unipi.it

**Carosi Laura**
Dipartimento di Statistica e Matematica applicata all’Economia
Università di Pisa
Via Ridolfi, 10 56124 Pisa, Italy
lcarosi@ec.unipi.it

**Castagnoli Erio**
Dipartimento di Scienze delle Decisioni,
Università Commerciale “Luigi Bocconi”, via Röntgen 1,
20136 Milano, Italy
erio.castagnoli@uni-bocconi.it

**Castellani Marco**
Dipartimento Sistemi ed Istituzioni per l’Economia
Università degli Studi dell’Aquila,
Italy
castella@ec.univaq.it

**Giulì Massimiliano**
Dipartimento Sistemi ed Istituzioni per l’Economia
Università degli Studi dell’Aquila,
Italy
giuli@ec.univaq.it

**Favero Gino**
Dipartimento di Economia, Università degli Studi di Parma,
via Kennedy 6, 43100 Parma, Italy
gino.favero@unipr.it

**Giorgi Giorgetto**
Dipartimento di Economia e Metodi Quantitativi
Università degli studi di Pavia
Via S.Felice 5, Pavia, Italy
ggiorgi@eco.unipv.it

**Martein Laura**
Dipartimento di Statistica e Matematica applicata all’Economia
Università di Pisa
Via Ridolfi, 10 56124 Pisa, Italy
lmartein@ec.unipi.it
Mazzoleni Piera
Dipartimento di discipline matematiche, finanza matematica ed econometria, Università Cattolica del Sacro Cuore, Via Necchi 9, Milano, Italy
piera.mazzoleni@unicatt.it

Miglierina Enrico
Dipartimento di Economia
Università degli Studi dell'Insubria, via Monte Generoso 71, 21100 Varese, Italy
enrico.miglierina@uninsubria.it

Molho Elena
Dipartimento di Economia e Metodi Quantitativi
Università degli studi di Pavia
Via S.Felice 5, Pavia, Italy
molhoe@eco.unipv.it

Recchioni Maria Cristina
Dipartimento di Scienze Sociali “D. Serrani”, Università Politecnica delle Marche, Piazza Martelli 8, 60121 Ancona, Italy
mcrecchi@mta01.univpm.it

Ricardi Rossana
Dipartimento di Statistica e Matematica applicata all’Economia
Università di Pisa
Via Ridolfi, 10 56124 Pisa, Italy
riccardi@ec.unipi.it

Siegfried Schaible
Department of Applied Mathematics, Chung Yuan Christian University, Chung-Li, Taiwan
e-mail: schaible2008@gmail.com

Sodini Claudio
Dipartimento di Statistica e Matematica applicata all’Economia
Università di Pisa
Via Ridolfi, 10 56124 Pisa, Italy
csodini@ec.unipi.it

Sodini Mauro
Dipartimento di Statistica e Matematica applicata all’Economia
Università di Pisa
Via Ridolfi, 10 56124 Pisa, Italy
m.sodini@ec.unipi.it
Efficiency under pseudoinvexity, and duality in differentiable and non-differentiable multiobjective problems. A characterization

Arana-Jiménez, Manuel *

Departamento de Estadística e I.O., Escuela Superior de Ingeniería, Universidad de Cádiz, 11002 Cádiz, Spain.
e-mail: manuel.arana@uca.es

Summary. In this work, characterizations for efficient solutions in non-differentiable multiobjective programming problems are commented from current researching by the author, which generalize recent characterization results for differentiable multi-objective programming problems. So, in order for Kuhn-Tucker points to be efficient solutions it is both necessary and sufficient that the multiobjective problem functions belong to a new class of functions, called KT-pseudoinvex-II. An example is proposed to illustrate this class of function and optimality results. A dual problem is studied and weak, strong and converse duality results are presented.

Keywords: Multiobjective programming, invexity, pseudoinvexity, optimality conditions, efficient solutions.

1 Introduction and Preliminaries

This work is based on recent definitions and results in differentiable multiobjective mathematical programming provided by Arana et al.\cite{1, 2}, as well as their current researching in the non-differentiable case. So, this work is focused on the search for efficient solutions to mathematical programming problems through the study of optimality conditions and of the properties of the functions that are involved, as well as through the study of dual problems.

For scalar problems, in the case of optimality conditions, it is customary to use critical points of the Kuhn-Tucker or Fritz-John \cite{3} types. In the case of the kinds of functions employed in mathematical programming problems, the tendency has been to substitute convex functions with more general ones, with the objective of obtaining a solution through an optimality condition. Meanwhile, the inverse result has also sometimes been sought.

Let us consider the following unconstrained scalar problem:

* This work was partially supported by the grant MTM2007-063432 of the Science and Education Spanish Ministry.
Minimize $\theta(x)$
subject to:

$x \in S \subseteq \mathbb{R}^n$

where $\theta : S \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ is a differentiable function on the open set $S$. It is known that if $\bar{x}$ is a solution, then $\bar{x}$ is a stationary point. It is an interesting problem to look for the class of functions for which the reciprocal holds, that is, in which a critical or stationary point $\bar{x}$ is necessarily a solution of $(P)$. In this way, the class of invex functions introduced by Craven and Hanson (see [4, 5]) closed the problem. Let us remind these definitions and results it.

**Definition 1.** Let $\theta : S \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ be a differentiable function on the open set $S$. Then, $\theta$ is said to be invex if there exists a vector function $\eta : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that $\forall x, \bar{x} \in S$

$$\theta(x) - \theta(\bar{x}) \geq \nabla \theta(\bar{x}) \eta(x, \bar{x}).$$

The class of invex functions was extended to the class of pseudoinvex functions.

**Definition 2.** Let $\theta : S \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ be a differentiable function on the open set $S$. Then, $\theta$ is said to be pseudoinvex if there exists a vector function $\eta : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that $\forall x, \bar{x} \in S$

$$\theta(x) - \theta(\bar{x}) < 0 \Rightarrow \nabla \theta(\bar{x}) \eta(x, \bar{x}) < 0.$$

However, these classes of functions are equivalent, (see Ben-Israel and Mond [6]). With the introduction of invex function, the equivalence between a solution of a scalar problem $(P)$ and a stationary point was established; furthermore, this characterizes the invex functions as the next theorem states [6].

**Theorem 1.** $\theta$ is an invex function if and only if all stationary points are optimal solutions of $(P)$.

Several authors have generalized convexity and invexity, and they have continued the study of optimal solutions for constrained scalar problems ([7], [8], [9],...), formulated as follows:

\[(CP)\] Minimize $\theta(x)$
subject to:

$$g(x) \leq 0$$

$$x \in S \subseteq \mathbb{R}^n$$

with $\theta : S \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$, $g = (g_1, \ldots, g_m) : S \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^m$, $\theta, g_1, \ldots, g_m$ are differentiable functions on the open set $S \subseteq \mathbb{R}^n$. 
For constrained scalar problems (CP), invexity is a sufficient condition for which a Kuhn-Tucker critical point leads to a solution of (CP), but it is not necessary. Martin [10] defined a weaker concept, called KT-invexity, and proved that it is a necessary and sufficient condition in order for a Kuhn-Tucker critical point to be an optimal solution of (CP).

**Definition 3.** Problem (CP) is said to be KT-invex if there exists a function \( \eta : S \times S \to \mathbb{R}^n \) such that \( \forall x, \bar{x} \in S, \) with \( g_i(x) \leq 0, g_i(\bar{x}) \leq 0, i = 1, \ldots, m, \)
\[
\theta(x) - \theta(\bar{x}) \geq \nabla \theta(\bar{x}) \eta(x, \bar{x}),
\]
\[-\nabla g_j(\bar{x}) \eta(x, \bar{x}) \geq 0, \quad \forall j \in I(\bar{x}),
\]
where
\[ I(\bar{x}) = \{ j : j = 1, \ldots, m \text{ such that } g_j(\bar{x}) = 0 \}. \]

Martin [10] obtained the following result:

**Theorem 2.** Every Kuhn-Tucker critical point is an optimal solution of (CP) if and only if (CP) is KT-invex.

The outline is to present classes of functions that make up unconstrained and constrained multiobjective problems, such that any class of functions which is characterized by having every Kuhn-Tucker critical point as an efficient solution must be equivalent to these classes of functions. So, invexity, pseudoinvexity [4, 5] and KT-invexity [10] functions, as well as their characterizations results, are generalized. In sections 2 and 3, new classes of vector functions, based on generalized invexity, are introduced for the study of efficient solutions for differentiable multiobjective programming problems, and the study of duality. Moreover, an example is proposed to illustrate these new classes of vector functions and results obtained. In section 4, an extension to the non-differentiable case is proposed. Finally, in section 5, some comments on author current research are made.

**2 Differentiable unconstrained problem. Efficiency**

Let us introduce the following convention for equalities and inequalities.

If \( x = (x_1, \ldots, x_n), \) \( y = (y_1, \ldots, y_n) \in \mathbb{R}^n, \) then
\[
x = y \Leftrightarrow x_i = y_i, \quad \forall i = 1, \ldots, n,
\]
\[ x < y \Leftrightarrow x_i < y_i, \quad \forall i = 1, \ldots, n, \]
\[ x \leq y \Leftrightarrow x_i \leq y_i, \quad \forall i = 1, \ldots, n, \]
\[ x \leq y \Leftrightarrow x_i \leq y_i, \quad \forall i = 1, \ldots, n, \quad \text{and there exists } j \text{ such that } x_j < y_j. \]

Similarly, \( >, \geq, \leq. \) Firstly, let us center on unconstrained multiobjective problems, which, in general, can be formulated as follows:
Minimize \( f(x) = (f_1(x), \ldots, f_p(x)) \)
subject to:
\[ x \in S \subseteq \mathbb{R}^n \]
where \( f : S \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^p \) is differentiable and \( S \) is said to be the feasible set. The solutions of (MP) are named efficient points and they were introduced by Pareto [11].

**Definition 4.** A feasible point, \( \bar{x} \), is said to be an efficient solution of (MP) if there does not exist another feasible point, \( x \), such that \( f(x) \leq f(\bar{x}) \).

Later, it appeared a more general concept such as the weakly efficient solution of (MP).

**Definition 5.** A feasible point, \( \bar{x} \), is said to be a weakly efficient solution of (MP) if there does not exist another feasible point, \( x \), such that \( f(x) < f(\bar{x}) \).

It is easy to see that any efficient point is a weakly efficient point. The following optimality condition type is usually employed.

**Definition 6.** A feasible point for (MP), \( \bar{x} \), is said to be a vector critical point if there exists \( \lambda \in \mathbb{R}^p \), such that
\[ \lambda^T \nabla f(\bar{x}) = 0, \]
\[ \lambda \geq 0, \]
where \( \nabla f(\bar{x}) \in M^{p \times n} \) is the gradient matrix of the vector function \( f \), and \( M^{p \times n} \) denotes the set of \( p \times n \) real matrices. The vector critical point condition is necessary for a feasible point of (MP) to be efficient solution (see [12, 13]) or a weakly efficient solution (see [14]), as the next theorem states.

**Theorem 3.** If \( \bar{x} \) is a weakly efficient solution for (MP), then \( \bar{x} \) is a vector critical point.

Now, the objective is to extend the result in Theorem 1 to multiobjective programming. For that, let us consider the following definition, which extends the concept of scalar invex function to the multiple case (see [15, 16]).

**Definition 7.** Let \( f = (f_1, \ldots, f_p) : S \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^p \) be a differentiable function on the open set \( S \). Then, the vector function \( f \) is said to be invex if there exists a vector function \( \eta : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n \) such that \( \forall x, \bar{x} \in S \)
\[ f(x) - f(\bar{x}) \geq \nabla f(\bar{x})\eta(x, \bar{x}) \]
Osuna et al. [15], and recently Arana et al. [1] define two classes of vector functions generalizing the class of scalar pseudoinvex functions.
Definition 8. Let \( f = (f_1, \ldots, f_p) : S \subseteq \mathbb{R}^n \to \mathbb{R}^p \) be a differentiable function on the open set \( S \). Then the vector function \( f \) is said to be pseudoinvex-I if there exists a vector function \( \eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \) such that \( \forall x, \bar{x} \in S \)

\[
    f(x) - f(\bar{x}) < 0 \implies \nabla f(\bar{x}) \eta(x, \bar{x}) < 0.
\]

Definition 9. Let \( f = (f_1, \ldots, f_p) : S \subseteq \mathbb{R}^n \to \mathbb{R}^p \) be a differentiable function on the open set \( S \). Then the vector function \( f \) is said to be pseudoinvex-II if there exists a vector function \( \eta : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \) such that \( \forall x, \bar{x} \in S \)

\[
    f(x) - f(\bar{x}) \leq 0 \implies \nabla f(\bar{x}) \eta(x, \bar{x}) < 0.
\]

In the scalar case, i.e., when \( p = 1 \), Definitions 7, 8 and 9 are equivalent (see [6]).

The vector critical point condition is a necessary optimality condition for a point to be an efficient or weakly efficient solution of the multiobjective problem (MP), (see [5, 14]). Osuna et al. [15] proved that pseudoinvexity-I is necessary and sufficient for the set of vector critical points and the set of weakly efficient solutions of (MP) to be equivalent.

Theorem 4. Every vector critical point is a weakly efficient solution of (MP) if and only if \( f \) is pseudoinvex-I.

Recently, Arana et al.[1] have extended this result to the study of efficient solutions, as follows:

Theorem 5. Every critical point is an efficient solution of (MP) if and only if \( f \) is pseudoinvex-II.

This result generalizes Theorem 1 to the multiple case. Furthermore, the equivalence between vector critical points and efficient solutions of (MP) characterizes the class of pseudoinvex-II vector functions.

We have seen that pseudoinvexity plays an important role in (MP). In the scalar case, the classes of invex, pseudoinvex-I and pseudoinvex-II functions are equivalent. However, these are different classes of functions in the vectorial case, such as the following theorem establishes [1].

\[
    PSI = \{ f : S \subseteq \mathbb{R}^n \to \mathbb{R}^p / f \text{ is pseudoinvex-I} \},
    PSII = \{ f : S \subseteq \mathbb{R}^n \to \mathbb{R}^p / f \text{ is pseudoinvex-II} \},
    INV = \{ f : S \subseteq \mathbb{R}^n \to \mathbb{R}^p / f \text{ is invex} \}.
\]

Theorem 6. \( INV \cup PSII \subseteq PSI \).

Consequently, the relationship between invex, pseudoinvex-I and pseudoinvex-II functions is as follows:
3 Differentiable constrained problem. Duality

Now, let us add some constraints to our multiobjective problem, and we have:

\[(MP) \quad \text{Minimize } f(x) = (f_1(x), \ldots, f_p(x))\]
subject to:
\[x \in S \subseteq \mathbb{R}^n\]
where \(g = (g_1, \ldots, g_m) : S \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^m\) is differentiable. Let denote \(K\) the feasible set. In a similar way as (CP), Kuhn-Tucker conditions allow us to obtain efficient solutions for (CMP).

**Definition 10.** A feasible point \(\bar{x}\) for (CMP) is said to be a Kuhn-Tucker vector critical point, KTVCP, if there exist \(\lambda \in \mathbb{R}^p, \mu \in \mathbb{R}^m\) such that

\[
\lambda^T \nabla f(\bar{x}) + \mu^T \nabla g(\bar{x}) = 0 \quad (1)
\]
\[
\mu^T g(\bar{x}) = 0 \quad (2)
\]
\[
\mu \geq 0 \quad (3)
\]
\[
\lambda \geq 0 \quad (4)
\]

Based on results by Chankong and Haimes [17], Kanniappan [12], and Gulati and Talaat [13], the following Kuhn-Tucker optimality result is obtained for efficient solutions of (CMP), for which we need to take on a constraint qualification.

**Theorem 7.** If \(\bar{x}\) is an efficient solution of (CMP) and a constraint qualification is satisfied at \(\bar{x}\), then \(\bar{x}\) is a KTVCP.

For the study of efficient points of (CMP) from these Kuhn-Tucker optimality conditions, Arana et al.[2] have introduced new kind of vector function and problem.