HELICOIDAL SHELL THEORY

by

Teodoro Merlini and Marco Morandini

Scientific Report

DIA-SR 08-07

2008

Politecnico di Milano
Dipartimento di Ingegneria Aerospaziale

ARACNE

www.aracneeditrice.it
info@aracneeditrice.it

via Raffaele Garofalo, 133 A/B
00173 Roma
(06) 93781065

ISBN 978–88–548–1889–7

1st edition: July 2008

Contents

Contents **5**

1 Introduction **7**

2 Shell kinematical model **8**

2.1 Geometry of the shell **8**
2.1.1 Reference systems 8
2.1.2 Oriento-position 8
2.1.3 Base vectors 9
2.1.4 Curvature 9

2.2 Constant curvature motion of the material surface **10**
2.2.1 Geometry of the material surface 10
2.2.2 Constant rototranslation speed 10
2.2.3 Tangent curvature dual vectors 10
2.2.4 Oriento-position speed dual vector 11
2.2.5 Constant curvature motion 11

2.3 The helicoidal shell model **12**
2.3.1 The oriento-position field 12
2.3.2 Model curvatures 13
2.3.3 Model base vectors 14
2.3.4 Model curvature tensor 15
2.3.5 Model surface and volume elements 15
2.3.6 Relaxation of the kinematical hypotheses 16
2.3.7 Initialization of the dual director variable 17
2.3.8 Geometric invariance 17

2.4 Deformed configuration **17**
2.4.1 Deformed model oriento-position 18
2.4.2 Deformed model curvature 18
2.4.3 Deformed model base vectors and deformation gradient 19

2.5 Rototranslation of the shell model **19**
2.5.1 The rototranslation field 19
2.5.2 Shell model strain 20
2.5.3 The kinematical strain 20

2.6 Perturbed configuration **21**
2.6.1 Co-rototranslational variations of the model kinematical strain 21

3 Shell model mechanics **22**

3.1 Shell model equations **22**

3.2 Variational formulation **23**
3.2.1 Shell model variational framework 23
3.2.2 Boundary terms and boundary constraints 24
3.2.3 Principle of Virtual Work 25

3.3 Consistent linearization **25**
3.3.1 Internal stress contribution 25

4 Shell model local constitutive law 27

4.1 Nonlinear elasticity 27
4.1.1 Transverse strain local variable 28
4.1.2 Transverse strain local condensation 31
4.1.3 Transverse kinematical strain restraint 33

4.2 Linear elasticity 35

4.3 Linearized virtual functional 37

5 Reduction to the shell surface mechanics 38

5.1 Shell Biot-axial model 38
5.1.1 Substitute Biot-axial field 38
5.1.2 Biot-axial field variations 39

5.2 Surface density of the internal stress virtual functional 39
5.2.1 Shell model internal stress virtual functional 39
5.2.2 Material surface internal stress virtual functional 41
5.2.3 Recovering the transverse strain local variable 42

5.3 Shell constitutive mechanism 42
5.3.1 Proposition 42
5.3.2 Linearized Principle of Virtual Work of the shell 43
5.3.3 Shell director and Biot-axial local variables 44
5.3.4 Shell constitutive equations and tangent map 45
5.3.5 Alternative choices of the Biot-axial local variables 48

Appendices 51

A1 Kinematical strain variations 52

A1.1 Curvature co-rototranslational variation formulae 52
A1.1.1 Setting the co-rototranslational variation formulae 52
A1.1.2 Reducing the co-rototranslational variation formulae 56

A1.2 Kinematical strain co-rototranslational variation formulae 61

A2 Hyperelastic constitutive laws 63

A2.1 Three-dimensional models for non-polar isotropic materials 63
A2.1.1 The Saint Venant-Kirchhoff constitutive model 63
A2.1.2 The Neo-Hookean constitutive model 64
A2.1.3 Saint Venant-Kirchhoff constitutive model by Biot-type parameterization 65
A2.1.4 A linear constitutive law by Biot-type parameterization 66
A2.1.5 Neo-Hookean constitutive model by Biot-type parameterization 67
A2.1.6 Pseudo-Hookean constitutive model by Biot-type parameterization 68
A2.1.7 Synopsis 70

A2.2 Shell model local constitutive laws for non-polar isotropic materials 71
A2.2.1 Nonlinear constitutive laws 71
A2.2.2 Linear constitutive law 72
A2.2.3 Linear constitutive law using technical constants 76

A3 Internal stress virtual functional 80

A3.1 Surface density of the internal stress virtual functional 80
A3.1.1 Shell model internal stress virtual functional 80
A3.1.2 Material surface internal stress virtual functional 87
A3.1.3 Recovering the transverse strain local variable 88

References 91

1 Introduction

The present report attains the second step in the undertaking of introducing the *helicoidal modeling* (Merlini and Morandini, 2004a, 2004c) in computational shell mechanics. The main variational formulations of the *shell material surface* in the context of the helicoidal modeling have been developed in a first report by Merlini (2008b). That report focused on the intrinsic mechanics of the shell described by a two-coordinate manifold, whereas in the present second report, we proceed to recover the *solid shell* constitutive mechanism from three-dimensional continuum mechanics within a helicoidal modeling context.

The theory presented herein is composed of the solid shell kinematical model and the relevant variational mechanics. The shell is regarded as a three-dimensional continuum made of infinitesimal yet three-dimensional material particles, susceptible of a polar description. A kinematical model of this solid shell is a substitute three-dimensional body made of exactly the same material particles, but whose oriento-positions are described as known functions of a discrete set of parameters across the thickness (a kind of approximation close to the finite element method). The kinematical model we propose stands on very simple hypotheses in the context of the helicoidal modeling of the continuum: *the curvature dual vectors*, that characterize the oriento-position gradient, *are assumed constant across the shell thickness*. As a result, the model configuration depends on the configuration of the underlying material surface (Merlini, 2008b), and on one *dual director* that characterizes the generalized curvature across the thickness.

The variational mechanics of the solid shell model is nothing but an application of the variational formulation for three-dimensional continuum mechanics (Merlini and Morandini, 2004c, 2005). For simplicity, only a hyperelastic non-polar medium is considered, and just the 'one-field' formulation – the internally constrained Principle of Virtual Work – is written and consistently linearized. To represent the axial of the Biot stress within the shell model, a suitable and simple substitute field is hypothesized: *the local Biot-axial is assumed linear across the thickness* and depending on two parameters that are conveniently connected to the *dual Biot-axial* field of the underlying material surface (Merlini, 2008b). Once the kinematical and the Biot-axial models are substituted within the variational formulation and the appropriate integrals have been computed across the thickness, the Principle of Virtual Work must correspond to that obtained for the material surface. A further unknown is present now, however, besides the oriento-position and dual Biot-axial of the material surface: the dual director of the shell model.

The shell theory so far outlined yields the constitutive law to exploit in the variational mechanics of the material surface. The constitutive mechanism descends from an obvious statement of energetic equivalence between the virtual internal work formulated for the solid shell model and that of the material surface. This statement stands on a nonlinear virtual functional and is extended to its linearized form, which is continuously rebuilt during the incremental solution process. Since the dual director unknown does not pertain to the material surface mechanics, it would be convenient to solve it within the constitutive characterization itself, by means of a local condensation technique dynamically operated on the linearized tangent map. This condensation process needs a constraint equation and an unknown reaction to be accomplished, and the angular part of the surface dual Biot-axial is used for this purpose, leaving a surface mechanics theory endowed with only the oriento-position and the linear part of the surface dual Biot-axial as unknowns. The result of the constitutive mechanism is the expression of the stress resultants (forces and couples per unit length) and the relevant derivatives with respect to the kinematical strains of the material surface, namely the shell *constitutive equations* and their *tangent* map. Such quantities are integrals over the shell thickness, numerically obtained during the iterative solution process.

The Report is organized as follows. In Section 2 we introduce and discuss the constant-curvatures shell kinematical model. In Section 3 we reformulate the three-dimensional variational mechanics for the solid shell model and state and linearize the Principle of Virtual Work. In Section 4 we accommodate the local constitutive law for the particular shell model kinematics. In Section 5 we focus on the internal work linearized functional: we propose a substitute Biot-axial field and perform the integrals over the shell thickness; then, the shell constitutive equations and their tangent map are formulated. In the appendices we give details of the derivation of the kinematical strain variations (Appendix A1), of some basic local hyperelastic models (Appendix A2), and of the linearization of the internal stress virtual functionals (Appendix A3).

Notation

Throughout this Report, Latin indexes denote generic tensor components in the three-dimensional space, so they are intended to vary in the range 1 to 3. Instead, Greek indexes denote components relating to the shell two-dimensional surface and are intended to vary in the range 1 to 2.

The Einsteinian rule of implicit summation over repeated indexes is usually understood.

2 Shell kinematical model

As *solid shell*, we understand a three-dimensional body, lying on a smooth curved surface and thin in the direction locally normal to the surface. The thickness of this three-dimensional shell body is not necessarily constant over the surface; however, it is allowed to vary in a quite smooth manner so that the solid shell can always be referred to as a thin solid.

In agreement with the polar description of material solids, the solid shell is defined as a continuum set of infinitesimal yet three-dimensional material particles occupying the volume of a three-dimensional shell body. The status of each particle is identified by its position and by the orientation of an embedded triad of independent vectors. In a deformable shell, the material particles may change their relative distances and rotations, yet they keep on belonging to a deformed three-dimensional shell body, again lying on a smooth curved surface and thin in the direction locally normal to the surface.

In this chapter, we propose a kinematical model of the solid shell based on the *helicoidal modeling* of the continuum. Although this model is an approximation of the actual shell kinematics, it provides a natural description — suitable for curved thin bodies — that can be advantageously exploited in computational shell mechanics. The description of the solid shell is formulated here as an extension of the *material surface* description, as established in Merlini (2008b). The reader should refer to Chapter 2 of that Report as the fundament of the present shell kinematical model.

2.1 Geometry of the shell

The geometry of the solid shell is formulated in exactly the same way as for any three-dimensional solid (Merlini and Morandini, 2002, 2004c). In this section, we recall the essential steps of the formulation in the context of the helicoidal modeling.

2.1.1 Reference systems

First, we assume an absolute reference frame. We take as *absolute frame* an orthonormal triad of dimensionless unit vectors $\boldsymbol{i}_j \equiv \boldsymbol{i}^j$, fixed and located somewhere in the three-dimensional space, at a point called the absolute origin. This absolute frame, though arbitrary, is completely known, in both position and orientation, to anybody. The absolute frame is represented by the identity real tensor $\boldsymbol{I} = \boldsymbol{i}_j \otimes \boldsymbol{i}^j$.

Associated with this frame, we conveniently define an absolute system of coordinates. The *absolute reference system* is based on Cartesian coordinates measured along three orthogonal axes x^j issued from the absolute frame at the origin and oriented as the unit vectors $\boldsymbol{i}_j$, respectively. This reference system is implicitly assumed throughout the present theory; in fact, for numerical purposes, every variable in a formulation of continuum mechanics shall be ultimately measured in a unique, absolute system.

For the theoretical developments, we also resort to an intrinsic reference system made of convective coordinates. This *material reference system* is based on (dimensionless) curvilinear coordinates measured along three families of smooth lines ξ^j traced within the three-dimensional shell body. As usual, the coordinates ξ^α (α = 1, 2) are assumed to lie on surfaces 'parallel' to the shell faces, in the sense that they never cross the lower and upper faces of the solid shell. Instead, the coordinate lines ξ^3 do cross them. The coordinate surfaces (ξ^1, ξ^2) will be often referred to as *layers*. The lower and upper shell faces are coordinate surfaces themselves, located at some abscissas ξ^{3-} and ξ^{3+}, respectively.

2.1.2 Oriento-position

A material particle of the solid shell is identified by its position vector $\boldsymbol{x}$ and by the orientation tensor $\boldsymbol{\alpha} = \boldsymbol{\alpha}_j \otimes \boldsymbol{i}^j$ of a triad of vectors $\boldsymbol{\alpha}_j$ embedded in it. This triad must be unique for all the particles except for a relative rotation, so the particle orientations can also be understood as rotations from a unique triad. Although it is not strictly necessary, we assume that this unique triad is the absolute orthonormal frame itself: by doing so, the orientation tensors $\boldsymbol{\alpha}$ are actually rotation tensors.

A dual frame, as defined in Merlini and Morandini (2004c), is a complex geometrical entity suited to identify at the same time the position and the orientation of a material particle. This leads to a pole-based description and entails the choice of a pole. If we make the pole to coincide with the origin itself, a material particle is identified by the origin-based *oriento-position* dual tensor