BOUNDARY CONSTRAINT VARIATIONAL FORMULATION FOR HELICOIDAL MODELING

by

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Contents

1 Introduction

In the recent works on variational formulations and finite-element approximations for the helicoidal modeling in finite elasticity (Merlini and Morandini, 2002, 2004b, 2004c, 2004d, 2005), the treatment of the boundary constraints in a weak way was left incomplete. The main reason for this deficiency was the intrinsically bad consistency of the helicoidal modeling by itself with selective constraints. In fact, selective constraints, as found in customary practice, are mostly written in self-based kinematics rather than helicoidal kinematics.

The present Report aims at introducing a variational formulation of the boundary constraints suitable to manage selective constraints and at bringing it to work in a pole-based, helicoidal modeling finite-element context. This task is accomplished as follows. First, a constraint virtual functional is derived within the classical Euclidean-rotational modeling context and consistently linearized (Section 2.1). Next, a constraint frame is introduced to allow for the simplest case of selective constraints along three orthogonal directions (Section 2.2). In Section 3, after establishing the proper relationship between helicoidal and Euclidean-rotational kinematics, the constraint virtual functional is rewritten in terms of helicoidal variation variables. Finally, a simple and versatile boundary-constraint finite element, the node-constraint, is formulated in Section 4.

This report also gives us the opportunity to offer a revised edition of the early variational formulations for finite elasticity endowed with an independent rotation field (Merlini, 1995, 1997). Appendix A holds a thorough presentation of these formulations that someway extends well beyond the purpose of this report and can be addressed as a stand-alone document.

2 Constraint virtual functional

The boundary surface constraints are dealt with in this report in a weak manner. The aim of this section is to derive a virtual functional suitable for a weak form of the constraints, and to operate a consistent linearization on it. The sought virtual functional is intended to be used within a weaker-type variational principle, like the Principle of Hu-Washizu or the Principle of Hellinger-Reissner.

In this section, we completely adhere to a variational formulation for finite elasticity endowed with the classical Euclidean displacement field and with an uncoupled, *independent rotation field*. Variational formulations characterized by an independent rotation field have been discussed extensively in Merlini (1995, 1997). For the reader's convenience, the full, early formulation is concisely rewritten below as Appendix A. That straightforward formulation is exhaustive for both the polar and the non-polar continuum.

The weak-form constraint formulation we draw in this section is ready for the general polar continuum. In the case of a non-polar continuum, we knew that no external surface couples are locally allowed, so no rotation constraints can be locally prescribed at the boundary surface, see Box [A.5.1-7]. Nevertheless, in the approximation context of a variational principle, e.g. by the finite element method, the boundary surface equations are satisfied in a weak manner, hence scalar equations accounting for local external surface couples and local rotation constraints are allowed for. Therefore, the constraint virtual functional we will draw in this section can be applied as is to the non-polar continuum too.

2.1 Standard formulation

The standard formulation refers to the case of boundary particles fully constrained in both position and orientation. The case of selective constraints will be dealt with in Section 2.2.

2.1.1 Constrained-boundary equations and virtual functional

The equilibrium and compatibility equations at the boundary surface of a polar continuum were introduced in Section 0, and can be taken from Box [A.3.1-1]. It is convenient to logically split the boundary surface S into a free/loaded portion S_f and a constrained portion S_c (i.e. $S = S_\mathrm{f} \cup S_\mathrm{c}$). Our interest is focused on the latter portion.

At the constrained boundary surface S_c, the equilibrium equations

$$\boldsymbol{\Phi}\hat{\boldsymbol{T}}\boldsymbol{\nu} = \boldsymbol{t}_\mathrm{c} \qquad \boldsymbol{\Phi}\hat{\boldsymbol{M}}\boldsymbol{\nu} = \boldsymbol{m}_\mathrm{c} \tag{2-1}$$

can be assumed as fulfilled and define the *unknown reaction* densities $\boldsymbol{t}_\mathrm{c}$ and $\boldsymbol{m}_\mathrm{c}$. Instead, the compatibility equations

$$\boldsymbol{u} = \boldsymbol{u}_\mathrm{c} \qquad \boldsymbol{\varphi} = \boldsymbol{\varphi}_\mathrm{c}\,, \tag{2-2}$$

with $\boldsymbol{u}_\mathrm{c}$ and $\boldsymbol{\varphi}_\mathrm{c}$ known displacements and rotations, are retained as the *constraint equations* to fulfill.

In a variational context, the proper virtual multipliers of Eqs. (2-1) and (2-2) can be chosen as in Box [A.3.1-3], so the scalar form of Eqs. (2-1) and (2-2) writes (remember that $\boldsymbol{\Gamma} = \boldsymbol{\Gamma}^\mathrm{T}\boldsymbol{\Phi}$)

$$\begin{cases} \langle \delta\boldsymbol{u}, -\boldsymbol{\Phi}\hat{\boldsymbol{T}}\boldsymbol{\nu} + \boldsymbol{t}_\mathrm{c}\rangle + \langle \boldsymbol{\varphi}_\delta, -\boldsymbol{\Phi}\hat{\boldsymbol{M}}\boldsymbol{\nu} + \boldsymbol{m}_\mathrm{c}\rangle = 0 \\ \langle \delta\boldsymbol{t}_\mathrm{c}, \boldsymbol{u} - \boldsymbol{u}_\mathrm{c}\rangle + \langle \delta(\boldsymbol{\Gamma}^\mathrm{T}\boldsymbol{m}_\mathrm{c}), \boldsymbol{\varphi} - \boldsymbol{\varphi}_\mathrm{c}\rangle = 0 \end{cases} \tag{2-3}$$

Eqs. (2-3) are used in the derivation of the variational principles (see Sections A.3 and A.4), and the main weaker-form principles are obtained in Section A.4.2. The *virtual functional* relevant to the *constrained boundary* can be worked out as follows (remember that $\boldsymbol{\varphi}_\delta = \boldsymbol{\Gamma}\delta\boldsymbol{\varphi}$),

$$\begin{aligned} &\Pi_{\delta S_\mathrm{c}} \\ &\quad = -\int_{S_\mathrm{c}} \Big(\langle \delta\boldsymbol{u}, \boldsymbol{t}_\mathrm{c}\rangle + \langle \boldsymbol{\varphi}_\delta, \boldsymbol{m}_\mathrm{c}\rangle + \langle \delta\boldsymbol{t}_\mathrm{c}, \boldsymbol{u} - \boldsymbol{u}_\mathrm{c}\rangle + \langle \delta(\boldsymbol{\Gamma}^\mathrm{T}\boldsymbol{m}_\mathrm{c}), \boldsymbol{\varphi} - \boldsymbol{\varphi}_\mathrm{c}\rangle \Big)\,\mathrm{d}S_\mathrm{c} \\ &\quad = -\int_{S_\mathrm{c}} \Big(\delta\langle \boldsymbol{t}_\mathrm{c}, \boldsymbol{u} - \boldsymbol{u}_\mathrm{c}\rangle + \delta\langle \boldsymbol{\Gamma}^\mathrm{T}\boldsymbol{m}_\mathrm{c}, \boldsymbol{\varphi} - \boldsymbol{\varphi}_\mathrm{c}\rangle + \cancel{\langle \boldsymbol{m}_\mathrm{c}, (\boldsymbol{\varphi}_\delta - \boldsymbol{\Gamma}\delta(\boldsymbol{\varphi} - \boldsymbol{\varphi}_\mathrm{c}))\rangle} \Big)\,\mathrm{d}S_\mathrm{c} \end{aligned}$$

and it is finally written in the form

$$\Pi_{\delta S_c} = -\int_{S_c} \delta\left[\langle \boldsymbol{t}_c, \boldsymbol{u}-\boldsymbol{u}_c\rangle + \langle \boldsymbol{\Gamma}^{\mathrm{T}}\boldsymbol{m}_c, \boldsymbol{\varphi}-\boldsymbol{\varphi}_c\rangle\right]\mathrm{d}S_c \ . \tag{2-4}$$

So, it corresponds to the virtual variation of a finite functional, $\Pi_{S_c} = -\int_{S_c}\left(\langle \boldsymbol{t}_c, \boldsymbol{u}-\boldsymbol{u}_c\rangle + \langle \boldsymbol{\Gamma}^{\mathrm{T}}\boldsymbol{m}_c, \boldsymbol{\varphi}-\boldsymbol{\varphi}_c\rangle\right)\mathrm{d}S_c$.

2.1.2 Virtual functional linearization

The virtual functional $\Pi_{\delta S_c}$ is developed and linearized in this section. We will exploit the following formulae, taken from Merlini (2002, 2003):

$$\partial\boldsymbol{\varphi}_\delta = \boldsymbol{\varphi}_{\partial\delta} + \tfrac{1}{2}\boldsymbol{\varphi}_\partial \times \boldsymbol{\varphi}_\delta \tag{2-5}$$

$$\begin{aligned} \delta\boldsymbol{\Gamma} &= \boldsymbol{\Gamma}_{/} : \delta\boldsymbol{\varphi}\otimes\boldsymbol{I} \\ \delta\boldsymbol{\Gamma}_{/} &= \boldsymbol{\Gamma}_{//}^{\underline{12}\underline{34}} : \delta\boldsymbol{\varphi}\otimes\boldsymbol{I} \end{aligned} \tag{2-6}$$

See Merlini (2003) for the expression of tensors $\boldsymbol{\Gamma}$, $\boldsymbol{\Gamma}_{/}$ and $\boldsymbol{\Gamma}_{//}^{\underline{12}\underline{34}}$.

$$\begin{aligned} \Pi_{\delta S_c} &= -\int_{S_c} \delta\left[\langle \boldsymbol{t}_c, \boldsymbol{u}-\boldsymbol{u}_c\rangle + \langle \boldsymbol{\Gamma}^{\mathrm{T}}\boldsymbol{m}_c, \boldsymbol{\varphi}-\boldsymbol{\varphi}_c\rangle\right]\mathrm{d}S_c \\ &= -\int_{S_c}\left(\langle \boldsymbol{t}_c, \delta(\boldsymbol{u}-\boldsymbol{u}_c)\rangle + \langle \boldsymbol{m}_c, \delta(\boldsymbol{\Gamma}(\boldsymbol{\varphi}-\boldsymbol{\varphi}_c))\rangle + \langle \delta\boldsymbol{t}_c, \boldsymbol{u}-\boldsymbol{u}_c\rangle + \langle \delta\boldsymbol{m}_c, \boldsymbol{\Gamma}(\boldsymbol{\varphi}-\boldsymbol{\varphi}_c)\rangle\right)\mathrm{d}S_c \\ &= -\int_{S_c}\left(\langle \delta\boldsymbol{u}, \boldsymbol{t}_c\rangle + \langle \boldsymbol{\Gamma}\delta(\boldsymbol{\varphi}-\boldsymbol{\varphi}_c) + \delta\boldsymbol{\Gamma}(\boldsymbol{\varphi}-\boldsymbol{\varphi}_c), \boldsymbol{m}_c\rangle + \langle \delta\boldsymbol{t}_c, \boldsymbol{u}-\boldsymbol{u}_c\rangle + \langle \delta\boldsymbol{m}_c, \boldsymbol{\Gamma}(\boldsymbol{\varphi}-\boldsymbol{\varphi}_c)\rangle\right)\mathrm{d}S_c \\ &= -\int_{S_c}\left(\langle \delta\boldsymbol{u}, \boldsymbol{t}_c\rangle + \langle \boldsymbol{\Gamma}\delta\boldsymbol{\varphi} + \boldsymbol{\Gamma}_{/} : \delta\boldsymbol{\varphi}\otimes(\boldsymbol{\varphi}-\boldsymbol{\varphi}_c), \boldsymbol{m}_c\rangle + \langle \delta\boldsymbol{t}_c, \boldsymbol{u}-\boldsymbol{u}_c\rangle + \langle \delta\boldsymbol{m}_c, \boldsymbol{\Gamma}(\boldsymbol{\varphi}-\boldsymbol{\varphi}_c)\rangle\right)\mathrm{d}S_c \\ &= -\int_{S_c}\left(\langle \delta\boldsymbol{u}, \boldsymbol{t}_c\rangle + \langle (\boldsymbol{\Gamma} + \boldsymbol{\Gamma}_{/}(\boldsymbol{\varphi}-\boldsymbol{\varphi}_c))\boldsymbol{\Gamma}^{-1}\boldsymbol{\varphi}_\delta, \boldsymbol{m}_c\rangle + \langle \delta\boldsymbol{t}_c, \boldsymbol{u}-\boldsymbol{u}_c\rangle + \langle \delta\boldsymbol{m}_c, \boldsymbol{\Gamma}(\boldsymbol{\varphi}-\boldsymbol{\varphi}_c)\rangle\right)\mathrm{d}S_c \\ &= -\int_{S_c}\left(\langle \delta\boldsymbol{u}, \boldsymbol{t}_c\rangle + \langle \boldsymbol{\varphi}_\delta, (\boldsymbol{I} + \boldsymbol{\Gamma}^{-\mathrm{T}}\boldsymbol{\Gamma}_{/}^{\mathrm{T}213}(\boldsymbol{\varphi}-\boldsymbol{\varphi}_c))\boldsymbol{m}_c\rangle + \langle \delta\boldsymbol{t}_c, \boldsymbol{u}-\boldsymbol{u}_c\rangle + \langle \delta\boldsymbol{m}_c, \boldsymbol{\Gamma}(\boldsymbol{\varphi}-\boldsymbol{\varphi}_c)\rangle\right)\mathrm{d}S_c \end{aligned}$$

$$\begin{aligned} \partial\Pi_{\delta S_c} &= -\int_{S_c}\left(\partial\langle \delta\boldsymbol{u}, \boldsymbol{t}_c\rangle + \partial\langle \boldsymbol{\varphi}_\delta, (\boldsymbol{I} + \boldsymbol{\Gamma}^{-\mathrm{T}}\boldsymbol{\Gamma}_{/}^{\mathrm{T}213}(\boldsymbol{\varphi}-\boldsymbol{\varphi}_c))\boldsymbol{m}_c\rangle + \partial\langle \delta\boldsymbol{t}_c, \boldsymbol{u}-\boldsymbol{u}_c\rangle + \partial\langle \delta\boldsymbol{m}_c, \boldsymbol{\Gamma}(\boldsymbol{\varphi}-\boldsymbol{\varphi}_c)\rangle\right)\mathrm{d}S_c \\ &= -\int_{S_c}\Big(\langle \partial\delta\boldsymbol{u}, \boldsymbol{t}_c\rangle + \langle \partial\boldsymbol{\varphi}_\delta, (\boldsymbol{I} + \boldsymbol{\Gamma}^{-\mathrm{T}}\boldsymbol{\Gamma}_{/}^{\mathrm{T}213}(\boldsymbol{\varphi}-\boldsymbol{\varphi}_c))\boldsymbol{m}_c\rangle + \langle \boldsymbol{\varphi}_\delta, \partial(\boldsymbol{I} + \boldsymbol{\Gamma}^{-\mathrm{T}}\boldsymbol{\Gamma}_{/}^{\mathrm{T}213}(\boldsymbol{\varphi}-\boldsymbol{\varphi}_c))\boldsymbol{m}_c\rangle \\ &\qquad + \langle \delta\boldsymbol{u}, \partial\boldsymbol{t}_c\rangle + \langle \boldsymbol{\varphi}_\delta, (\boldsymbol{I} + \boldsymbol{\Gamma}^{-\mathrm{T}}\boldsymbol{\Gamma}_{/}^{\mathrm{T}213}(\boldsymbol{\varphi}-\boldsymbol{\varphi}_c))\partial\boldsymbol{m}_c\rangle \\ &\qquad + \langle \partial\delta\boldsymbol{t}_c, \boldsymbol{u}-\boldsymbol{u}_c\rangle + \langle \partial\delta\boldsymbol{m}_c, \boldsymbol{\Gamma}(\boldsymbol{\varphi}-\boldsymbol{\varphi}_c)\rangle + \langle \delta\boldsymbol{t}_c, \partial(\boldsymbol{u}-\boldsymbol{u}_c)\rangle + \langle \delta\boldsymbol{m}_c, \partial(\boldsymbol{\Gamma}(\boldsymbol{\varphi}-\boldsymbol{\varphi}_c))\rangle\Big)\mathrm{d}S_c \\ &= -\int_{S_c}\Big(\langle \partial\delta\boldsymbol{u}, \boldsymbol{t}_c\rangle + \langle \partial\boldsymbol{\varphi}_\delta, (\boldsymbol{I} + \boldsymbol{\Gamma}^{-\mathrm{T}}\boldsymbol{\Gamma}_{/}^{\mathrm{T}213}(\boldsymbol{\varphi}-\boldsymbol{\varphi}_c))\boldsymbol{m}_c\rangle \\ &\qquad + \langle \boldsymbol{\varphi}_\delta, \left(\boldsymbol{\Gamma}^{-\mathrm{T}}\boldsymbol{\Gamma}_{/}^{\mathrm{T}213}\partial\boldsymbol{\varphi} + \boldsymbol{\Gamma}^{-\mathrm{T}}\partial\boldsymbol{\Gamma}_{/}^{\mathrm{T}213}(\boldsymbol{\varphi}-\boldsymbol{\varphi}_c) - \boldsymbol{\Gamma}^{-\mathrm{T}}\partial\boldsymbol{\Gamma}^{\mathrm{T}}\boldsymbol{\Gamma}^{-\mathrm{T}}\boldsymbol{\Gamma}_{/}^{\mathrm{T}213}(\boldsymbol{\varphi}-\boldsymbol{\varphi}_c)\right)\boldsymbol{m}_c\rangle \\ &\qquad + \langle \delta\boldsymbol{u}, \partial\boldsymbol{t}_c\rangle + \langle \boldsymbol{\varphi}_\delta, (\boldsymbol{I} + \boldsymbol{\Gamma}^{-\mathrm{T}}\boldsymbol{\Gamma}_{/}^{\mathrm{T}213}(\boldsymbol{\varphi}-\boldsymbol{\varphi}_c))\partial\boldsymbol{m}_c\rangle \\ &\qquad + \langle \partial\delta\boldsymbol{t}_c, \boldsymbol{u}-\boldsymbol{u}_c\rangle + \langle \partial\delta\boldsymbol{m}_c, \boldsymbol{\Gamma}(\boldsymbol{\varphi}-\boldsymbol{\varphi}_c)\rangle + \langle \delta\boldsymbol{t}_c, \partial\boldsymbol{u}\rangle + \langle \delta\boldsymbol{m}_c, \boldsymbol{\Gamma}\partial\boldsymbol{\varphi} + \partial\boldsymbol{\Gamma}(\boldsymbol{\varphi}-\boldsymbol{\varphi}_c)\rangle\Big)\mathrm{d}S_c \\ &= -\int_{S_c}\Big(\langle \partial\delta\boldsymbol{u}, \boldsymbol{t}_c\rangle + \langle \partial\boldsymbol{\varphi}_\delta, (\boldsymbol{I} + \boldsymbol{\Gamma}^{-\mathrm{T}}\boldsymbol{\Gamma}_{/}^{\mathrm{T}213}(\boldsymbol{\varphi}-\boldsymbol{\varphi}_c))\boldsymbol{m}_c\rangle \\ &\qquad + \langle \boldsymbol{\varphi}_\delta, \boldsymbol{\Gamma}^{-\mathrm{T}}\left(\boldsymbol{\Gamma}_{/}\partial\boldsymbol{\varphi} + (\boldsymbol{\Gamma}_{//}^{\underline{12}\underline{34}} : \partial\boldsymbol{\varphi}\otimes\boldsymbol{I})(\boldsymbol{\varphi}-\boldsymbol{\varphi}_c) - (\boldsymbol{\Gamma}_{/}(\boldsymbol{\varphi}-\boldsymbol{\varphi}_c))\boldsymbol{\Gamma}^{-1}(\boldsymbol{\Gamma}_{/} : \partial\boldsymbol{\varphi}\otimes\boldsymbol{I})\right)^{\mathrm{T}}\boldsymbol{m}_c\rangle \\ &\qquad + \langle \delta\boldsymbol{u}, \partial\boldsymbol{t}_c\rangle + \langle \boldsymbol{\varphi}_\delta, (\boldsymbol{I} + \boldsymbol{\Gamma}^{-\mathrm{T}}\boldsymbol{\Gamma}_{/}^{\mathrm{T}213}(\boldsymbol{\varphi}-\boldsymbol{\varphi}_c))\partial\boldsymbol{m}_c\rangle \\ &\qquad + \langle \partial\delta\boldsymbol{t}_c, \boldsymbol{u}-\boldsymbol{u}_c\rangle + \langle \partial\delta\boldsymbol{m}_c, \boldsymbol{\Gamma}(\boldsymbol{\varphi}-\boldsymbol{\varphi}_c)\rangle + \langle \delta\boldsymbol{t}_c, \partial\boldsymbol{u}\rangle + \langle \delta\boldsymbol{m}_c, \boldsymbol{\Gamma}\partial\boldsymbol{\varphi} + \boldsymbol{\Gamma}_{/} : \partial\boldsymbol{\varphi}\otimes(\boldsymbol{\varphi}-\boldsymbol{\varphi}_c)\rangle\Big)\mathrm{d}S_c \end{aligned}$$

$$
\begin{aligned}
&= -\int_{S_\mathrm{c}} \Big(\langle \partial\delta\boldsymbol{u}, \boldsymbol{t}_\mathrm{c} \rangle + \langle \boldsymbol{\varphi}_{\partial\delta} + \tfrac{1}{2}\boldsymbol{\varphi}_{\partial} \times \boldsymbol{\varphi}_{\delta}, (\boldsymbol{I} + \boldsymbol{\Gamma}^{-\mathrm{T}}\boldsymbol{\Gamma}_{/}^{\mathrm{T}213}(\boldsymbol{\varphi}-\boldsymbol{\varphi}_\mathrm{c}))\boldsymbol{m}_\mathrm{c} \rangle \\
&\qquad + \langle \boldsymbol{\varphi}_{\delta}, \boldsymbol{\Gamma}^{-\mathrm{T}} \Big(\boldsymbol{m}_\mathrm{c} \cdot \Big(\boldsymbol{\Gamma}_{/} + \boldsymbol{\Gamma}_{//}^{1\underline{23}4}(\boldsymbol{\varphi}-\boldsymbol{\varphi}_\mathrm{c}) - (\boldsymbol{\Gamma}_{/}^{\mathrm{T}132}\boldsymbol{\Gamma}^{-1}\boldsymbol{\Gamma}_{/}^{\mathrm{T}132})^{\mathrm{T}1342}(\boldsymbol{\varphi}-\boldsymbol{\varphi}_\mathrm{c}) \Big) \partial\boldsymbol{\varphi} \Big) \rangle \\
&\qquad + \langle \delta\boldsymbol{u}, \partial\boldsymbol{t}_\mathrm{c} \rangle + \langle \boldsymbol{\varphi}_{\delta}, (\boldsymbol{I} + \boldsymbol{\Gamma}^{-\mathrm{T}}\boldsymbol{\Gamma}_{/}^{\mathrm{T}213}(\boldsymbol{\varphi}-\boldsymbol{\varphi}_\mathrm{c}))\partial\boldsymbol{m}_\mathrm{c} \rangle \\
&\qquad + \langle \partial\delta\boldsymbol{t}_\mathrm{c}, \boldsymbol{u}-\boldsymbol{u}_\mathrm{c} \rangle + \langle \partial\delta\boldsymbol{m}_\mathrm{c}, \boldsymbol{\Gamma}(\boldsymbol{\varphi}-\boldsymbol{\varphi}_\mathrm{c}) \rangle + \langle \delta\boldsymbol{t}_\mathrm{c}, \partial\boldsymbol{u} \rangle + \langle \delta\boldsymbol{m}_\mathrm{c}, \boldsymbol{\Gamma}\partial\boldsymbol{\varphi} + (\boldsymbol{\Gamma}_{/}(\boldsymbol{\varphi}-\boldsymbol{\varphi}_\mathrm{c}))\partial\boldsymbol{\varphi} \rangle \Big) \mathrm{d}S_\mathrm{c} \\
&= -\int_{S_\mathrm{c}} \Big(\langle \partial\delta\boldsymbol{u}, \boldsymbol{t}_\mathrm{c} \rangle + \langle \boldsymbol{\varphi}_{\partial\delta}, (\boldsymbol{I} + \boldsymbol{\Gamma}^{-\mathrm{T}}\boldsymbol{\Gamma}_{/}^{\mathrm{T}213}(\boldsymbol{\varphi}-\boldsymbol{\varphi}_\mathrm{c}))\boldsymbol{m}_\mathrm{c} \rangle + \langle \boldsymbol{\varphi}_{\delta}, \tfrac{1}{2}((\boldsymbol{I} + \boldsymbol{\Gamma}^{-\mathrm{T}}\boldsymbol{\Gamma}_{/}^{\mathrm{T}213}(\boldsymbol{\varphi}-\boldsymbol{\varphi}_\mathrm{c}))\boldsymbol{m}_\mathrm{c}) \times \boldsymbol{\varphi}_{\partial} \rangle \\
&\qquad + \langle \boldsymbol{\varphi}_{\delta}, \boldsymbol{\Gamma}^{-\mathrm{T}} \Big(\boldsymbol{m}_\mathrm{c} \cdot \Big(\boldsymbol{\Gamma}_{/} + (\boldsymbol{\Gamma}_{//}^{1\underline{23}4} - (\boldsymbol{\Gamma}_{/}^{\mathrm{T}132}\boldsymbol{\Gamma}^{-1}\boldsymbol{\Gamma}_{/}^{\mathrm{T}132})^{\mathrm{T}1342})\,(\boldsymbol{\varphi}-\boldsymbol{\varphi}_\mathrm{c}) \Big) \Big) \boldsymbol{\Gamma}^{-1}\boldsymbol{\varphi}_{\partial} \rangle \\
&\qquad + \langle \delta\boldsymbol{u}, \partial\boldsymbol{t}_\mathrm{c} \rangle + \langle \boldsymbol{\varphi}_{\delta}, (\boldsymbol{I} + \boldsymbol{\Gamma}^{-\mathrm{T}}\boldsymbol{\Gamma}_{/}^{\mathrm{T}213}(\boldsymbol{\varphi}-\boldsymbol{\varphi}_\mathrm{c}))\partial\boldsymbol{m}_\mathrm{c} \rangle \\
&\qquad + \langle \partial\delta\boldsymbol{t}_\mathrm{c}, \boldsymbol{u}-\boldsymbol{u}_\mathrm{c} \rangle + \langle \partial\delta\boldsymbol{m}_\mathrm{c}, \boldsymbol{\Gamma}(\boldsymbol{\varphi}-\boldsymbol{\varphi}_\mathrm{c}) \rangle + \langle \delta\boldsymbol{t}_\mathrm{c}, \partial\boldsymbol{u} \rangle + \langle \delta\boldsymbol{m}_\mathrm{c}, (\boldsymbol{I} + \boldsymbol{\Gamma}^{-\mathrm{T}}\boldsymbol{\Gamma}_{/}^{\mathrm{T}213}(\boldsymbol{\varphi}-\boldsymbol{\varphi}_\mathrm{c}))^{\mathrm{T}}\boldsymbol{\varphi}_{\partial} \rangle \Big) \mathrm{d}S_\mathrm{c}
\end{aligned}
$$

After introducing, just for convenience, four tensor functions of the constraint equations,

$$
\begin{aligned}
\boldsymbol{Q}_{\mathrm{Lc}} &= \boldsymbol{u} - \boldsymbol{u}_\mathrm{c} \\
\boldsymbol{Q}_{\mathrm{Ic}} &= \boldsymbol{\Gamma}(\boldsymbol{\varphi}-\boldsymbol{\varphi}_\mathrm{c}) \\
\boldsymbol{Q}_{\mathrm{IIc}} &= \boldsymbol{I} + \boldsymbol{\Gamma}^{-\mathrm{T}}\boldsymbol{\Gamma}_{/}^{\mathrm{T}213}(\boldsymbol{\varphi}-\boldsymbol{\varphi}_\mathrm{c}) \\
\boldsymbol{Q}_{\mathrm{IIIc}} &= \boldsymbol{\Gamma}_{/} + (\boldsymbol{\Gamma}_{//}^{1\underline{23}4} - (\boldsymbol{\Gamma}_{/}^{\mathrm{T}132}\boldsymbol{\Gamma}^{-1}\boldsymbol{\Gamma}_{/}^{\mathrm{T}132})^{\mathrm{T}1342})\,(\boldsymbol{\varphi}-\boldsymbol{\varphi}_\mathrm{c})
\end{aligned}
\tag{2-7}
$$

the functional $\Pi_{\delta S_\mathrm{c}}$ and its linearized form $\partial\Pi_{\delta S_\mathrm{c}}$ are finally written as

$$
\begin{aligned}
\Pi_{\delta S_\mathrm{c}} &= -\int_{S_\mathrm{c}} \Big(\langle \delta\boldsymbol{u}, \boldsymbol{t}_\mathrm{c} \rangle + \langle \boldsymbol{\varphi}_{\delta}, \boldsymbol{Q}_{\mathrm{IIc}}\boldsymbol{m}_\mathrm{c} \rangle + \langle \delta\boldsymbol{t}_\mathrm{c}, \boldsymbol{Q}_{\mathrm{Lc}} \rangle + \langle \delta\boldsymbol{m}_\mathrm{c}, \boldsymbol{Q}_{\mathrm{Ic}} \rangle \Big) \mathrm{d}S_\mathrm{c} \\
\partial\Pi_{\delta S_\mathrm{c}} &= -\int_{S_\mathrm{c}} \Big(\langle \partial\delta\boldsymbol{u}, \boldsymbol{t}_\mathrm{c} \rangle + \langle \boldsymbol{\varphi}_{\partial\delta}, \boldsymbol{Q}_{\mathrm{IIc}}\boldsymbol{m}_\mathrm{c} \rangle + \langle \partial\delta\boldsymbol{t}_\mathrm{c}, \boldsymbol{Q}_{\mathrm{Lc}} \rangle + \langle \partial\delta\boldsymbol{m}_\mathrm{c}, \boldsymbol{Q}_{\mathrm{Ic}} \rangle \\
&\qquad + \langle \boldsymbol{\varphi}_{\delta}, (\boldsymbol{\Gamma}^{-\mathrm{T}}(\boldsymbol{m}_\mathrm{c} \cdot \boldsymbol{Q}_{\mathrm{IIIc}})\boldsymbol{\Gamma}^{-1} + \tfrac{1}{2}(\boldsymbol{Q}_{\mathrm{IIc}}\boldsymbol{m}_\mathrm{c})\times)\boldsymbol{\varphi}_{\partial} \rangle \\
&\qquad + \langle \delta\boldsymbol{u}, \partial\boldsymbol{t}_\mathrm{c} \rangle + \langle \boldsymbol{\varphi}_{\delta}, \boldsymbol{Q}_{\mathrm{IIc}}\partial\boldsymbol{m}_\mathrm{c} \rangle + \langle \delta\boldsymbol{t}_\mathrm{c}, \partial\boldsymbol{u} \rangle + \langle \delta\boldsymbol{m}_\mathrm{c}, \boldsymbol{Q}_{\mathrm{IIc}}^{\mathrm{T}}\boldsymbol{\varphi}_{\partial} \rangle \Big) \mathrm{d}S_\mathrm{c}
\end{aligned}
\tag{2-8}
$$

2.2 Selective constraints

Often, in structural analysis, we are concerned with problems where material points are constrained along some directions while they keep free along other directions (we say that the constraint is *selective*). In such cases, a boundary particle is constrained along a number of degrees-of-freedom out of *six*. Let's deal with the simplest case: the six d.o.f. of the boundary particle are *3 displacements* along the unit vectors of an *orthonormal local triad* and *3 rotations* around the same vectors.

2.2.1 Constraint frame

First, define the local *constraint frame*. Take three unit vectors $\boldsymbol{n}_j \equiv \boldsymbol{n}^j$, orthogonal to each other, and denote their orientation by the tensor

$$
\boldsymbol{N} = \boldsymbol{N}^{-\mathrm{T}} = \boldsymbol{n}_j \otimes \boldsymbol{i}^j = \boldsymbol{n}^j \otimes \boldsymbol{i}_j\,. \tag{2-9}
$$

The constraint frame $\boldsymbol{N}$ is assumed to be a given, known quantity.

Then, define the following dummy vectors,

$$
\begin{aligned}
\bar{\boldsymbol{u}}_\mathrm{c} &= \boldsymbol{N}^{\mathrm{T}}\boldsymbol{u}_\mathrm{c} & \qquad \bar{\boldsymbol{\varphi}}_\mathrm{c} &= \boldsymbol{N}^{\mathrm{T}}\boldsymbol{\varphi}_\mathrm{c} \\
\bar{\boldsymbol{t}}_\mathrm{c} &= \boldsymbol{N}^{\mathrm{T}}\boldsymbol{t}_\mathrm{c} & \qquad \bar{\boldsymbol{m}}_\mathrm{c} &= \boldsymbol{N}^{\mathrm{T}}\boldsymbol{m}_\mathrm{c}
\end{aligned}
\tag{2-10}
$$

Vectors $\bar{\boldsymbol{u}}_\mathrm{c}$ and $\bar{\boldsymbol{\varphi}}_\mathrm{c}$ are the known constraint displacements and rotation vectors back-rotated by $\boldsymbol{N}^{\mathrm{T}}$, and vectors $\bar{\boldsymbol{t}}_\mathrm{c}$ and $\bar{\boldsymbol{m}}_\mathrm{c}$ are the unknown constraint reactions back-rotated by $\boldsymbol{N}^{\mathrm{T}}$. Note that the absolute components of the dummy vectors are just the components of the true vectors in the local constraint frame. This fact makes it easy to apply selective constraints along a subset of the particle degrees-of-freedom. Just for convenience, also define the dummy vector unknowns

$$\bar{\boldsymbol{u}} = \boldsymbol{N}^{\mathrm{T}}\boldsymbol{u} \qquad \bar{\boldsymbol{\varphi}} = \boldsymbol{N}^{\mathrm{T}}\boldsymbol{\varphi} \,. \tag{2-11}$$

2.2.2 Linearized virtual functional

Introduce the following tensor functions of the constraint equations, which modify Eqs. (2-7):

$$\begin{aligned}
\bar{\boldsymbol{Q}}_{\mathrm{Lc}} &= \boldsymbol{N}^{\mathrm{T}}\boldsymbol{Q}_{\mathrm{Lc}} &&= \bar{\boldsymbol{u}} - \bar{\boldsymbol{u}}_{\mathrm{c}} \\
\bar{\boldsymbol{Q}}_{\mathrm{Ic}} &= \boldsymbol{N}^{\mathrm{T}}\boldsymbol{Q}_{\mathrm{Ic}} &&= \boldsymbol{N}^{\mathrm{T}}\boldsymbol{\Gamma}\boldsymbol{N}(\bar{\boldsymbol{\varphi}} - \bar{\boldsymbol{\varphi}}_{\mathrm{c}}) \\
\bar{\boldsymbol{Q}}_{\mathrm{IIc}} &= \boldsymbol{Q}_{\mathrm{IIc}}\boldsymbol{N} &&= \boldsymbol{N} + \boldsymbol{\Gamma}^{-\mathrm{T}}(\boldsymbol{N}^{\mathrm{T}}\boldsymbol{\Gamma}_{/}\boldsymbol{N}(\bar{\boldsymbol{\varphi}} - \bar{\boldsymbol{\varphi}}_{\mathrm{c}}))^{\mathrm{T}} \\
\bar{\boldsymbol{Q}}_{\mathrm{IIIc}} &= \boldsymbol{N}^{\mathrm{T}}\boldsymbol{Q}_{\mathrm{IIIc}} &&= \boldsymbol{N}^{\mathrm{T}}\boldsymbol{\Gamma}_{/} + \boldsymbol{N}^{\mathrm{T}}(\boldsymbol{\Gamma}_{//}^{1\underline{23}4} - (\boldsymbol{\Gamma}_{/}^{\mathrm{T}132}\boldsymbol{\Gamma}^{-1}\boldsymbol{\Gamma}_{/}^{\mathrm{T}132})^{\mathrm{T}1342})\, \boldsymbol{N}(\bar{\boldsymbol{\varphi}} - \bar{\boldsymbol{\varphi}}_{\mathrm{c}})
\end{aligned} \tag{2-12}$$

Using Eqs. (2-10)-(2-12), the functional $\Pi_{\delta S_{\mathrm{c}}}$ and its linearized form $\partial\Pi_{\delta S_{\mathrm{c}}}$ from Eqs. (2-8) become

$$\begin{aligned}
\Pi_{\delta S_{\mathrm{c}}} &= -\int_{S_{\mathrm{c}}}\Big(\langle\delta\boldsymbol{u}, \boldsymbol{N}\bar{\boldsymbol{t}}_{\mathrm{c}}\rangle + \langle\boldsymbol{\varphi}_{\delta}, \bar{\boldsymbol{Q}}_{\mathrm{IIc}}\bar{\boldsymbol{m}}_{\mathrm{c}}\rangle + \langle\delta\bar{\boldsymbol{t}}_{\mathrm{c}}, \bar{\boldsymbol{Q}}_{\mathrm{Lc}}\rangle + \langle\delta\bar{\boldsymbol{m}}_{\mathrm{c}}, \bar{\boldsymbol{Q}}_{\mathrm{Ic}}\rangle\Big)\mathrm{d}S_{\mathrm{c}} \\
\partial\Pi_{\delta S_{\mathrm{c}}} &= -\int_{S_{\mathrm{c}}}\Big(\langle\partial\delta\boldsymbol{u}, \boldsymbol{N}\bar{\boldsymbol{t}}_{\mathrm{c}}\rangle + \langle\boldsymbol{\varphi}_{\partial\delta}, \bar{\boldsymbol{Q}}_{\mathrm{IIc}}\bar{\boldsymbol{m}}_{\mathrm{c}}\rangle + \langle\partial\delta\bar{\boldsymbol{t}}_{\mathrm{c}}, \bar{\boldsymbol{Q}}_{\mathrm{Lc}}\rangle + \langle\partial\delta\bar{\boldsymbol{m}}_{\mathrm{c}}, \bar{\boldsymbol{Q}}_{\mathrm{Ic}}\rangle \\
&\qquad + \langle\boldsymbol{\varphi}_{\delta}, (\boldsymbol{\Gamma}^{-\mathrm{T}}(\bar{\boldsymbol{m}}_{\mathrm{c}}\cdot\bar{\boldsymbol{Q}}_{\mathrm{IIIc}})\boldsymbol{\Gamma}^{-1} + \tfrac{1}{2}(\bar{\boldsymbol{Q}}_{\mathrm{IIc}}\bar{\boldsymbol{m}}_{\mathrm{c}})\times)\boldsymbol{\varphi}_{\partial}\rangle \\
&\qquad + \langle\delta\boldsymbol{u}, \boldsymbol{N}\partial\bar{\boldsymbol{t}}_{\mathrm{c}}\rangle + \langle\boldsymbol{\varphi}_{\delta}, \bar{\boldsymbol{Q}}_{\mathrm{IIc}}\partial\bar{\boldsymbol{m}}_{\mathrm{c}}\rangle + \langle\delta\bar{\boldsymbol{t}}_{\mathrm{c}}, \boldsymbol{N}^{\mathrm{T}}\partial\boldsymbol{u}\rangle + \langle\delta\bar{\boldsymbol{m}}_{\mathrm{c}}, \bar{\boldsymbol{Q}}_{\mathrm{IIc}}^{\mathrm{T}}\boldsymbol{\varphi}_{\partial}\rangle\Big)\mathrm{d}S_{\mathrm{c}}
\end{aligned} \tag{2-13}$$

Functionals $\Pi_{\delta S_{\mathrm{c}}}$ and $\partial\Pi_{\delta S_{\mathrm{c}}}$ in Eqs. (2-13) can be freely substituted for Eqs. (2-8) also in the case of non-selective constraints. Note that Eqs. (2-8) are a particular case of Eqs. (2-13) with $\boldsymbol{N} \equiv \boldsymbol{I}$.

Eqs. (2-13) can be given the following matrix form:

$$\begin{aligned}
\Pi_{\delta S_{\mathrm{c}}} &= -\int_{S_{\mathrm{c}}}\left(\begin{Bmatrix}\boldsymbol{\varphi}_{\delta}\\ \delta\boldsymbol{u}\end{Bmatrix}^{\mathrm{T}}\cdot\begin{Bmatrix}\bar{\boldsymbol{Q}}_{\mathrm{IIc}}\bar{\boldsymbol{m}}_{\mathrm{c}}\\ \boldsymbol{N}\bar{\boldsymbol{t}}_{\mathrm{c}}\end{Bmatrix} + \begin{Bmatrix}\delta\bar{\boldsymbol{m}}_{\mathrm{c}}\\ \delta\bar{\boldsymbol{t}}_{\mathrm{c}}\end{Bmatrix}^{\mathrm{T}}\cdot\begin{Bmatrix}\bar{\boldsymbol{Q}}_{\mathrm{Ic}}\\ \bar{\boldsymbol{Q}}_{\mathrm{Lc}}\end{Bmatrix}\right)\mathrm{d}S_{\mathrm{c}} \\
\partial\Pi_{\delta S_{\mathrm{c}}} &= -\int_{S_{\mathrm{c}}}\left(\begin{Bmatrix}\boldsymbol{\varphi}_{\partial\delta}\\ \partial\delta\boldsymbol{u}\end{Bmatrix}^{\mathrm{T}}\cdot\begin{Bmatrix}\bar{\boldsymbol{Q}}_{\mathrm{IIc}}\bar{\boldsymbol{m}}_{\mathrm{c}}\\ \boldsymbol{N}\bar{\boldsymbol{t}}_{\mathrm{c}}\end{Bmatrix} + \begin{Bmatrix}\partial\delta\bar{\boldsymbol{m}}_{\mathrm{c}}\\ \partial\delta\bar{\boldsymbol{t}}_{\mathrm{c}}\end{Bmatrix}^{\mathrm{T}}\cdot\begin{Bmatrix}\bar{\boldsymbol{Q}}_{\mathrm{Ic}}\\ \bar{\boldsymbol{Q}}_{\mathrm{Lc}}\end{Bmatrix}\right. \\
&\qquad + \begin{Bmatrix}\boldsymbol{\varphi}_{\delta}\\ \delta\boldsymbol{u}\end{Bmatrix}^{\mathrm{T}}\cdot\begin{bmatrix}\boldsymbol{\Gamma}^{-\mathrm{T}}(\bar{\boldsymbol{m}}_{\mathrm{c}}\cdot\bar{\boldsymbol{Q}}_{\mathrm{IIIc}})\boldsymbol{\Gamma}^{-1} + \tfrac{1}{2}(\bar{\boldsymbol{Q}}_{\mathrm{IIc}}\bar{\boldsymbol{m}}_{\mathrm{c}})\times & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{0}\end{bmatrix}\cdot\begin{Bmatrix}\boldsymbol{\varphi}_{\partial}\\ \partial\boldsymbol{u}\end{Bmatrix} + \begin{Bmatrix}\boldsymbol{\varphi}_{\delta}\\ \delta\boldsymbol{u}\end{Bmatrix}^{\mathrm{T}}\cdot\begin{bmatrix}\bar{\boldsymbol{Q}}_{\mathrm{IIc}} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{N}\end{bmatrix}\cdot\begin{Bmatrix}\partial\bar{\boldsymbol{m}}_{\mathrm{c}}\\ \partial\bar{\boldsymbol{t}}_{\mathrm{c}}\end{Bmatrix} \\
&\qquad \left. + \begin{Bmatrix}\delta\bar{\boldsymbol{m}}_{\mathrm{c}}\\ \delta\bar{\boldsymbol{t}}_{\mathrm{c}}\end{Bmatrix}^{\mathrm{T}}\cdot\begin{bmatrix}\bar{\boldsymbol{Q}}_{\mathrm{IIc}}^{\mathrm{T}} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{N}^{\mathrm{T}}\end{bmatrix}\cdot\begin{Bmatrix}\boldsymbol{\varphi}_{\partial}\\ \partial\boldsymbol{u}\end{Bmatrix}\right)\mathrm{d}S_{\mathrm{c}}
\end{aligned} \tag{2-14}$$

3 Helicoidal modeling constraints

The *helicoidal modeling* of the continuum is an alternative representation that uses a complex rototranslation field of the material particles, where displacements and rotations are intrinsically coupled, see Merlini and Morandini (2002, 2004a, 2004b). The relevant variational formulation leads to weaker-type principles analog of the Hu-Washizu and Hellinger-Reissner principles, where the virtual functional relevant to the *constrained boundary* surface (Merlini and Morandini, 2004b) has an expression that can be worked out as follows,

$$
\begin{aligned}
&\mathit{\Pi}_{\delta S_\mathrm{c}} \\
&\quad = -\int_{S_\mathrm{c}} \Big(\langle \boldsymbol{\alpha}'_\delta, \boldsymbol{s}_\mathrm{c} \rangle + \langle \delta(\boldsymbol{\Lambda}^\mathrm{T} \boldsymbol{s}_\mathrm{c}), \operatorname{ax}\log \boldsymbol{H} - \operatorname{ax}\log \boldsymbol{H}_\mathrm{c} \rangle \Big) \mathrm{d}S_\mathrm{c} \\
&\quad = -\int_{S_\mathrm{c}} \Big(\langle \boldsymbol{\eta}_\delta, \boldsymbol{s}_\mathrm{c} \rangle + \langle \delta(\boldsymbol{\Lambda}^\mathrm{T} \boldsymbol{s}_\mathrm{c}), \boldsymbol{\eta} - \boldsymbol{\eta}_\mathrm{c} \rangle \Big) \mathrm{d}S_\mathrm{c} \\
&\quad = -\int_{S_\mathrm{c}} \Big(\delta\langle \boldsymbol{\Lambda}^\mathrm{T} \boldsymbol{s}_\mathrm{c}, \boldsymbol{\eta} - \boldsymbol{\eta}_\mathrm{c} \rangle + \cancel{\langle \boldsymbol{s}_\mathrm{c}, (\boldsymbol{\eta}_\delta - \boldsymbol{\Lambda}\delta(\boldsymbol{\eta} - \boldsymbol{\eta}_\mathrm{c})) \rangle} \Big) \mathrm{d}S_\mathrm{c}
\end{aligned}
$$

and finally written in the concise form

$$
\mathit{\Pi}_{\delta S_\mathrm{c}} = -\int_{S_\mathrm{c}} \delta\langle \boldsymbol{\Lambda}^\mathrm{T} \boldsymbol{s}_\mathrm{c}, \boldsymbol{\eta} - \boldsymbol{\eta}_\mathrm{c} \rangle \, \mathrm{d}S_\mathrm{c} \,. \tag{3-1}
$$

Eq. (3-1) could be worked out and linearized like Eq. (2-4), however it rarely or never would be suitable for the case of selective constraints. In fact, the single components of the constrained dual helix $\boldsymbol{\eta}$ lose in general their physical significance in practical usage, and, as pointed out by Merlini and Morandini (2004b), the direct treatment of selective constraints can be a difficult task in the helicoidal modeling context. The expedient is to use, instead of Eq. (3-1), the constrained-boundary virtual functional Eq. (2-4) of the classical Euclidean-rotational modeling also for the case of helicoidal modeling.

In this section we limit ourselves to accommodate the virtual functionals $\mathit{\Pi}_{\delta S_\mathrm{c}}$ and $\partial\mathit{\Pi}_{\delta S_\mathrm{c}}$ of the linearized form Eqs. (2-13) or (2-14) to the variational context proper of the helicoidal modeling.

3.1 Linearized virtual functional in terms of dual variables

The first step is to rewrite Eqs. (2-13) and (2-14) in terms of virtual and incremental dual vector variables. To this end, let's introduce the following kinematical dual vectors

$$
\begin{array}{llll}
\boldsymbol{e} & = \boldsymbol{\varphi} + \varepsilon\, \boldsymbol{u} & \bar{\boldsymbol{e}} = \bar{\boldsymbol{\varphi}} + \varepsilon\, \bar{\boldsymbol{u}} & = \boldsymbol{N}^\mathrm{T} \boldsymbol{e} \\
\boldsymbol{e}_\delta & = \boldsymbol{\varphi}_\delta + \varepsilon\, \delta\boldsymbol{u} & & \\
\boldsymbol{e}_\partial & = \boldsymbol{\varphi}_\partial + \varepsilon\, \partial\boldsymbol{u} & & \\
\boldsymbol{e}_{\partial\delta} & = \boldsymbol{\varphi}_{\partial\delta} + \varepsilon\, \partial\delta\boldsymbol{u} & & \\
\boldsymbol{e}_\mathrm{c} & = \boldsymbol{\varphi}_\mathrm{c} + \varepsilon\, \boldsymbol{u}_\mathrm{c} & \bar{\boldsymbol{e}}_\mathrm{c} = \bar{\boldsymbol{\varphi}}_\mathrm{c} + \varepsilon\, \bar{\boldsymbol{u}}_\mathrm{c} & = \boldsymbol{N}^\mathrm{T} \boldsymbol{e}_\mathrm{c}
\end{array}
\tag{3-2}
$$

and reaction dual vectors

$$
\begin{array}{llll}
\underleftarrow{\boldsymbol{s}}_\mathrm{c} & = \boldsymbol{t}_\mathrm{c} + \varepsilon\, \boldsymbol{m}_\mathrm{c} & \underleftarrow{\bar{\boldsymbol{s}}}_\mathrm{c} = \bar{\boldsymbol{t}}_\mathrm{c} + \varepsilon\, \bar{\boldsymbol{m}}_\mathrm{c} & = \boldsymbol{N}^\mathrm{T} \underleftarrow{\boldsymbol{s}}_\mathrm{c} \\
\delta\underleftarrow{\boldsymbol{s}}_\mathrm{c} & = \delta\boldsymbol{t}_\mathrm{c} + \varepsilon\, \delta\boldsymbol{m}_\mathrm{c} & \delta\underleftarrow{\bar{\boldsymbol{s}}}_\mathrm{c} = \delta\bar{\boldsymbol{t}}_\mathrm{c} + \varepsilon\, \delta\bar{\boldsymbol{m}}_\mathrm{c} & = \boldsymbol{N}^\mathrm{T} \delta\underleftarrow{\boldsymbol{s}}_\mathrm{c} \\
\partial\underleftarrow{\boldsymbol{s}}_\mathrm{c} & = \partial\boldsymbol{t}_\mathrm{c} + \varepsilon\, \partial\boldsymbol{m}_\mathrm{c} & \partial\underleftarrow{\bar{\boldsymbol{s}}}_\mathrm{c} = \partial\bar{\boldsymbol{t}}_\mathrm{c} + \varepsilon\, \partial\bar{\boldsymbol{m}}_\mathrm{c} & = \boldsymbol{N}^\mathrm{T} \partial\underleftarrow{\boldsymbol{s}}_\mathrm{c} \\
\partial\delta\underleftarrow{\boldsymbol{s}}_\mathrm{c} & = \partial\delta\boldsymbol{t}_\mathrm{c} + \varepsilon\, \partial\delta\boldsymbol{m}_\mathrm{c} & \partial\delta\underleftarrow{\bar{\boldsymbol{s}}}_\mathrm{c} = \partial\delta\bar{\boldsymbol{t}}_\mathrm{c} + \varepsilon\, \partial\delta\bar{\boldsymbol{m}}_\mathrm{c} & = \boldsymbol{N}^\mathrm{T} \partial\delta\underleftarrow{\boldsymbol{s}}_\mathrm{c}
\end{array}
\tag{3-3}
$$

Vectors in Eqs. (3-2) and (3-3) are self-based dual forms. Using such dual forms, $\mathit{\Pi}_{\delta S_\mathrm{c}}$ and $\partial\mathit{\Pi}_{\delta S_\mathrm{c}}$ of Eqs. (2-13) can be worked out as follows,