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I

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via Raffaele Garofalo, 133 A/B
00173 Roma
(06) 93781065

ISBN 978–88–548–0165–3

I edizione: dicembre 2008

Bollettino di Matematica pura e applicata

Volume I

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Generalized Weyl's theorem and quasi-affinity

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Abstract

A bounded operator $T \in L(X)$ acting on a Banach space X is said to satisfy generalized Weyl's theorem if the complement in the spectrum of the B-Weyl spectrum is the set of all eigenvalues which are isolated points of the spectrum. In this paper we prove that generalized Weyl's theorem holds for several classes of operators, extending previous results obtained in [23] and [14]. We also consider the preservation of generalized Weyl's theorem between two operators $T \in L(X)$, $S \in L(Y)$ in the case that these are intertwined by a quasi-affinity $A \in L(X, Y)$, or in the more general case that T and S are asymptotically intertwined by A.

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*Supported by Fondi ex-60, 2007, Università di Palermo.
†Supported by Protars D11/16 and PGR-UMP.

1 Introduction

If $T \in L(X)$, where $L(X)$ denote the Banach algebra of all bounded linear operators acting on a complex Banach space X, we denote by $N(T)$ its kernel and by $R(T)$ its range. The operator $T \in L(X)$ is called a *B-Fredholm* operator [8], if there is an integer n for which $R(T^n)$ is closed and such that the operator $T_n : R(T^n) \to R(T^n)$, defined by $T_n(x) = T(x)$ for every $x \in R(T^n)$, is a Fredholm operator. From [9, Theorem 3.1], it follows that T is a B-Fredholm operator if and only if there exists an integer n such that $c_n(T) < \infty$ and $c'_n(T) < \infty$, where $c_n(T) := \dim(R(T^n)/R(T^{n+1}))$, and $c'_n(T) := \dim(N(T^{n+1})/N(T^n))$. For a B-Fredholm operator $T \in L(X)$ the index of is defined by $ind(T) = c'_n(T) - c_n(T)$. From [8, Proposition 2.1], the index is independent of the choice of the integer n. Moreover, in the case of a Fredholm operator, this definition coincides with the classical definition of the index. Recall that the *ascent* of T is defined as $a(T) =: \inf\{n \in \mathbb{N} : N(T^n) = N(T^{n+1})\}$, while the *descent* is defined as $\delta(T) := \inf\{n \in \mathbb{N} : R(T^n) = R(T^{n+1})\}$ (in both cases the infimum over the empty set is taken ∞). An operator $T \in L(X)$ is said to be *left Drazin invertible* if $a(T) < \infty$ and $R(T^{a(T)+1})$ is closed. Note that if T has finite ascent and finite descent then $a(T) = \delta(T)$, see [1, Theorem 3.3]). It is well known that $0 < a(\lambda I - T) = \delta(\lambda I - T) < \infty$ precisely when λ is a pole of the resolvent of T. A Fredholm operator which have index 0 is called a *Weyl operator* and the Weyl spectrum is defined as $\sigma_W(T) = \{\lambda \in \mathbf{C} : T - \lambda I \text{ is not Weyl}\}$. These concept may be generalized ad follows:

Definition 1.1. *Let $T \in L(X)$. Then T is called a* B-Weyl operator *if it is a B-Fredholm operator of index* 0*. The* B-Weyl spectrum $\sigma_{BW}(T)$ *of T is defined by $\sigma_{BW}(T) = \{\lambda \in \mathbf{C} : T - \lambda I$ is not a B-Weyl operator$\}$.*

A classical result of Weyl on the fine structure of the spectrum of a normal operator T, defined on a Hilbert space, states that the equality $\sigma_W(T) = \sigma(T) \setminus E_0(T)$ holds, where $E_0(T)$ denotes the set of isolated eigenvalues of finite multiplicity. Successively, this result it has been extended to many other important classes of operators, and in literature T is said to satisfy *Weyl's theorem* if $\sigma_W(T) = \sigma(T) \setminus E_0(T)$. By [9, Theorem 4.5] any normal operator T also satisfies the equality $\sigma_{BW}(T) = \sigma(T) \setminus E(T)$, [9, Theorem 4.5], where $E(T)$ is the set of all eigenvalues of T which are isolated points of $\sigma(T)$, and in general $T \in L(X)$ is said to satisfy *generalized Weyl's theorem* if the equality $\sigma_{BW}(T) = \sigma(T) \setminus E(T)$ holds. Note that

$$\text{generalized Weyl's theorem} \Rightarrow \text{Weyl's theorem},$$

and the opposite implication is, in general, not true, see [12, Theorem 3.9].

Two weaker variants of Weyl's theorems are defined as follows: $T \in L(X)$ satisfies *Browder's theorem* if $\sigma_W(T) = \sigma(T) \setminus \Pi_0(T)$, where $\Pi_0(T)$ is the set of the poles $\lambda \in \mathbb{C}$ such that $T - \lambda$ is Fredholm (or equivalently Weyl, see [1, Theorem 3.4].) A bounded operator $T \in L(X)$ is said to satisfy *generalized Browder's* *theorem* if $\sigma_{BW}(T) = \sigma(T) \backslash \Pi(T)$, where $\Pi(T)$ denotes the set of all the poles of the resolvent of T. Note that, in general, the inclusion $\Pi(T) \subseteq E(T)$ holds for every $T \in L(X)$. Recently, it has been proved that Browder's theorems and generalized Browder's theorem for an operator are equivalent ([6]). Browder's theorem and Weyl's theorems are related as follows (see [10, Corollary 2.6] and [2, Theorem 3.1]):

Theorem 1.2. *Let $T \in L(X)$. Then T satisfies Weyl's theorem if and only if Browder's theorem holds for T and $E_0(T) = \Pi_0(T)$. Analogously, T satisfies the generalized Weyl's theorem if and only if generalized Browder's theorem (or, equivalently, Browder's theorem) holds for T and $E(T) = \Pi(T)$.*

In this paper we prove that generalized Weyl's theorem holds for several classes of operators, extending earlier results of [23] and [14]. In particular, we give sufficient and necessary conditions for a finitely ascensive operator in order to obey generalized Weyl's theorem. The preservation of Weyl's theorem, whenever $T \in L(X)$ and $S \in L(Y)$ are intertwined by a quasi-affinity, has been investigated in [24]. A part of this paper concerns the preservation of generalized Weyl's theorem for operators intertwined by a quasi-affinity. We shall also consider the more general case where T and S are asymptotically intertwined by a quasi-affinity.

2 Generalized Weyl's Theorem

The following useful lemma will be needed in the sequel.

Theorem 2.1. *If $T \in L(X)$ and $a(T) < \infty$ then the following statements are equivalent:*

(i) *There exists $n \geq a(T) + 1$ such that $T^n(X)$ is closed;*

(ii) *$T^n(X)$ is closed for all $n \geq p$.*

Proof. $k_i(T) := c'_i(T) - c'_{i+1}(T) = 0$ for all $i \geq a(T)$. The equivalence then easily follows from [28, Lemma 12]. □

Definition 2.2. We say that $T \in L(X)$ has the *single valued extension property* at $\lambda_0 \in \mathbb{C}$, if for an arbitrary open neighborhood U of λ_0, $f = 0$ is the only analytic function $f : U \to X$ such that $(T - \lambda I)f(\lambda) = 0$, for all $\lambda \in U$. We will say that T has the single valued extension property (SVEP) if T has this property at every $\lambda \in \mathbb{C}$.

Note that

$$a(T - \lambda I) < \infty \Rightarrow \ T \text{ has SVEP at } \lambda,$$

and the converse implication holds in the case that T is B-Fredholm, see [3].

An important subspace in local spectral theory is given by the *glocal spectral subspace* $\mathcal{X}_T(F)$ associated with a closed subset $F \subseteq \mathbb{C}$. This is defined, for an arbitrary operator $T \in L(X)$ and a closed subset F of $\mathbb{C}$, as the set of all $x \in X$ for which there exists an analytic function $f : \mathbb{C} \setminus F \to X$ which satisfies the identity

$$(\lambda I - T)f(\lambda) = x \quad \text{for all } \lambda \in \mathbb{C} \setminus F.$$

Recall that a bounded operator $T \in L(X)$, X a Banach space, is said to have *Dunford's property (C)*, shortly property (C), if, for each closed set $F \subseteq \mathbb{C}$, $\chi_T(F)$ is closed. It is well-known that Dunford property (C) implies SVEP.
If $T \in L(X)$, the *quasi-nilpotent part* of T is defined by $H_0(T) := \{x \in X : ||Tx^n||^{1/n} \to 0\}$. Note that $N(T^n) \subseteq H_0(T)$ for all $n \in \mathbb{N}$. Moreover, $H_0(T - \lambda I) = \mathcal{X}_T(\{\lambda\})$, see [1, Theorem 2.20]. The *analytical core* of T is defined $K(T) := \{x \in X :$ there exist $c > 0$ and a sequence $(x_n)_{n \geq 1} \subseteq X$ such that $Tx_1 = x, Tx_{n+1} = x_n$ for all $n \in \mathbb{N}$, and $||x_n|| \leq c^n||x||$for all $n \in \mathbb{N}\}$. Note that $T(K(T)) = K(T)$.
If λ is isolated in $\sigma(T)$, then it is known that $H_0(T - \lambda I)$ and $K(T - \lambda I)$ are closed subspaces of X, since $H_0(T - \lambda I)$ coincides with the range of the spectral projection P_λ associated with $\{\lambda\}$ and $K(T - \lambda I)$ coincides with the kernel of P_λ ([1, Theorem 3.74]). Therefore, $X = H_0(T - \lambda I) \oplus K(T - \lambda I)$, where $\oplus$ means the topological direct sum. In the case that λ is a pole, if $p := a(T - \lambda I) = \delta(T - \lambda I) < \infty$ then $H_0(T - \lambda I) = N(T - \lambda I)^p$ and $K(T - \lambda I) = R(T - \lambda I)^p$.
An operator T in $B(X)$ is *totally paranormal* if

$$||(T - \lambda I)x||^2 \leq ||(T - \lambda I)^2 x||||x||, \quad \text{for all } x \in X,\ \lambda \in \mathbf{C}.$$

Theorem 2.3. *Let T in $B(X)$ be a totally paranormal operator. Then the generalized Weyl's theorem holds for T.*

Proof. From [25] we know that if T is totally paranormal, then $T - \lambda I$ is of finite ascent for all $\lambda \in \mathbb{C}$. Hence T has SVEP. Also from [25, Corollary 4.8] , we have $H_0(T - \lambda I - T) = N(T - \lambda I)$ for all $\lambda \in \mathbb{C}$. From Theorem [11, Theorem 3.5], it then follows that T satisfies the generalized Weyl's theorem. $\square$

A spectral point $\lambda \in \sigma(T)$ is said to be a *bare point* if it lies on the circumference of some closed disc that contains $\sigma(T)$.

Theorem 2.4. *Suppose that for $T \in L(X)$ every $\lambda \in \sigma(T)$ is a bare point. Then T has SVEP*

Proof. Let $U \subseteq \mathbb{C}$ be an nonempty open set and let $f : U \to X$ be an analytic function such that $(T - \lambda I)f(\lambda) = 0$, for all $\lambda \in U$. Let $\rho(T) := \mathbb{C} \setminus \sigma(T)$ be the resolvent set of T. Since T satisfies condition β, then $U \cap \rho(T) \neq \emptyset$. Therefore there exists an open nonempty subset V contained in $U \cap \rho(T)$. Moreover for all $\lambda \in V$, we have $f(\lambda) = 0$, because $(T - \lambda I)$ is invertible. Since the zeros of a non vanishing analytic function are isolated, then $f = 0$. Hence T has SVEP. $\square$

Let $T \in L(X)$. We say that T satisfies the growth condition (G_m) if there exists an integer m such that

$$sup_{\lambda \notin \sigma(T)} ||(T - \lambda I)^{-1}|| dist(\lambda, \sigma(T))^m < \infty.$$

A bounded operator $T \in L(X)$ is said to be *isoloid* if every isolated point of the spectrum is an eigenvalue. In [15, Lemma 3] it is shown that every $T \in L(X)$ satisfies a growth condition (G_m) is *polaroid* (i.e. every isolated point of the spectrum is a pole) and hence isoloid. Let $\mathcal{H}(\sigma(T))$ denote the set of all analytic functions defined on an open neighborhood of $\sigma(T)$ and define, by the classical functional calculus, $f(T)$ for every $f \in \mathcal{H}(\sigma(T))$.

Corollary 2.5. *Suppose that $T \in L(X)$ has SVEP and satisfies a growth condition (G_m). Then $f(T)$ satisfies generalized Weyl's theorem for each $f \in \mathcal{H}(\sigma(T))$. In particular, if T satisfies a growth condition (G_m) and if every $\lambda \in \sigma(T)$ is a bare point then generalized Weyl's theorem holds for $f(T)$ for each $f \in \mathcal{H}(\sigma(T))$.*

Proof. Generalized Weyl's theorem for T is a direct consequence of Corollary [11, Corollary]. Since T satisfies SVEP the spectral theorem holds for $\sigma_{BW}(T)$. Moreover, T is isoloid so, by [14, Theorem 3.4], generalized Weyl's theorem holds for $f(T)$ for each $f \in \mathcal{H}(\sigma(T))$. The last statement is clear from Theorem 2.4. □

As noted above an operator satisfying generalized Weyl's theorem satisfies also Weyl's theorem. Corollary 2 extends an earlier result of Istratescu [23, Theorem], where Weyl's theorem was proved for bounded operators T for which the condition (G_1) holds for every restriction $T-\lambda I|M$ to every closed invariant subspace M and such that the bare point condition is satisfied. Observe that if the Banach space X is reflexive (in particular, a Hilbert space) the growth condition (G_m) for $T \in L(X)$ entails property (C), and consequently that T has SVEP [27, Proposition 1.2.19]. Since T is polaroid then $E(T) = \Pi(T)$, hence T satisfies generalized Weyl's theorem. The same argument of the proof of Corollary 2.5 then works also in this case, so $f(T)$ satisfies generalized Weyl's theorem, and in the case of reflexive Banach spaces, the bare point condition is not necessary.

Corollary 2.6. *Let $T \in L(X)$ be a spectral operator of finite type. Then $f(T)$ satisfies the generalized Weyl's theorem for each $f \in \mathcal{H}(\sigma(T))$.*

Proof. If $T \in L(H)$ is a spectral operator of finite type, then from [17] T satisfies a growth condition (G_m). Moreover, T has SVEP. □

Corollary 2.6 extends [29, Theorem 4] where it is proved that Weyl's theorem holds for a spectral operator of finite type.

An operator $T \in L(X)$ is said to be *reguloid* (see [22]) if for each isolated point $\lambda \in \sigma(T)$ the operator $T - \lambda I$ is regular in the sense that there exists an operator $S \in B(X)$ such that $T - \lambda I = (T - \lambda I)S(T - \lambda I)$.

Theorem 2.7. *If $T \in B(X)$ has the SVEP and if T is reguloid, then G-Weyl's theorem holds for T if and only if for all $\lambda \in E(T)$ the sequence $((N(T - \lambda I) \cap R((T - \lambda I)^n))_n$ is a stationary sequence for n large enough.*